

Heat Kernel Asymptotics, Path Integrals and Infinite-Dimensional Determinants

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Abstract

We investigate the short-time expansion of the heat kernel of a Laplace type operator on a compact Riemannian manifold and show that the lowest order term of this expansion is given by the Fredholm determinant of the Hessian of the energy functional on a space of finite energy paths. This is the asymptotic behavior to be expected from formally expressing the heat kernel as a path integral and then (again formally) using Laplace's method on the integral. We also relate this to the zeta determinant of the Jacobi operator, which is another way to assign a determinant to the Hessian of the energy functional. We consider both the near-diagonal asymptotics as well as the behavior at the cut locus.

1 Introduction and Main Results

Let M be a compact Riemannian manifold and let L be a formally self-adjoint Laplace type operator, acting on sections of a metric vector bundle \mathcal{V} over M . In this paper, we obtain results on the short-time asymptotics of the heat kernel $p_t^L(x, y)$ associated to L in terms of the geometry of the space of paths between x and y . Our results describe the quotient of $p_t^L(x, y)$ and the *Euclidean heat kernel*

$$e_t(x, y) := (4\pi t)^{-n/2} \exp\left(-\frac{1}{4t}d(x, y)^2\right), \quad (1.1)$$

which is the heat kernel of the Laplacian on functions in the case that M is the Euclidean space. In a sense, we describe the deviation of the metric of the manifold M from the flat metric in the vicinity of the minimizing geodesics between two points $x, y \in M$.

To explain the results, let $H_{xy}(M)$ be the space of absolutely continuous paths γ with $\gamma(0) = x$, $\gamma(1) = y$ and such that the velocity field satisfies $\dot{\gamma} \in L^2([0, 1], \gamma^*TM)$, i.e. the paths in $H_{xy}(M)$ have finite energy

$$E(\gamma) := \frac{1}{2} \int_0^1 |\dot{\gamma}(s)|^2 ds. \quad (1.2)$$

$H_{xy}(M)$ space is an infinite-dimensional manifold modelled on the Sobolev Hilbert space $H_0^1([0, 1], \mathbb{R}^n)$, and it carries the natural Riemannian metric

$$(X, Y)_{H^1} := \int_0^1 \langle \nabla_s X(s), \nabla_s Y(s) \rangle ds \quad (1.3)$$

that turns it into a Riemannian Hilbert manifold. Here we made the natural identification $T_\gamma H_{xy}(M) \cong H_0^1([0, 1], \gamma^*TM)$ for the tangent spaces.

Denote by Γ_{xy}^{\min} the set of length minimizing geodesics in $H_{xy}(M)$. We will always need the assumption that Γ_{xy}^{\min} is a submanifold of dimension k and that it is *non-degenerate* in the sense that for each $\gamma \in \Gamma_{xy}^{\min}$ the restriction $\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}$ of the Hessian of the energy to the normal space $N_\gamma \Gamma_{xy}^{\min}$ of Γ_{xy}^{\min} in $H_{xy}(M)$ is non-degenerate. Geometrically, this is equivalent to saying that for any Jacobi field X along a geodesic $\gamma \in \Gamma_{xy}^{\min}$ that vanishes at both endpoints, there exists a geodesic variation γ_s in direction X such that $\gamma_s \in H_{xy}(M)$ for each s . For example, this assumption is always satisfied when x and y are closer than the injectivity radius such that Γ_{xy}^{\min} only consists of the unique minimizing geodesic between the two points, and it also holds in the case that x and y are antipodal points of the round sphere S^n , in which case Γ_{xy}^{\min} is $(n-1)$ -dimensional.

Theorem 1.1. *Let L be a formally self-adjoint Laplace type operator acting on sections of a metric vector bundle \mathcal{V} over a compact Riemannian manifold M of dimension n . Suppose that for $x, y \in M$, the set of minimal geodesics Γ_{xy}^{\min} is a k -dimensional non-degenerate submanifold of the path space $H_{xy}(M)$. Then we have*

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{\Gamma_{xy}^{\min}} \frac{[\gamma|_0^1]^{-1}}{\det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma. \quad (1.4)$$

In particular, $\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}$ possesses a well-defined Fredholm determinant on the Hilbert space $N_\gamma \Gamma_{xy}^{\min}$ with respect to the metric (1.3).

Here the integrand on the right hand side involves the parallel transport $[\gamma|_0^1]$ determined by L : Any formally self-adjoint Laplace type operator can be written in the form $L = \nabla^* \nabla + V$ with a uniquely determined metric connection ∇ and a symmetric endomorphism field V , and it is this connection with respect to which we parallel transport in (1.4). In

(1.4), we integrate over Γ_{xy}^{\min} with respect to the Riemannian volume measure induced by the metric (1.3). Let us mention here that Γ_{xy}^{\min} is always compact by Prop. 2.4.11 in [Kli95] so that the integral on the right hand side of (1.4) is well-defined.

One can easily obtain a *formal proof* of Thm. 1.1 by using (in the case $L = \nabla^* \nabla$, let's say) that the heat kernel $p_t^L(x, y)$ is given formally by the path integral

$$p_t^L(x, y) \stackrel{\text{formally}}{=} (4\pi t)^{-n/2} \oint_{H_{xy}(M)} e^{-E(\gamma)/2t} [\gamma]_0^1{}^{-1} d^{H^1} \gamma. \quad (1.5)$$

where we would like to integrate with respect to the Riemannian volume measure on $H_{xy}(M)$ corresponding to the metric (1.3). The slash over the integral on the right hand sides denotes an additional division by $Z := (4\pi t)^{\dim(H_{xy}(M))/2}$, and here the problem with formula (1.5) becomes apparent: $H_{xy}(M)$ is infinite-dimensional, so the integral is multiplied by zero or infinity, depending on t . To make things worse, it is well known that there is no Lebesgue-style integration theory on $H_{xy}(M)$ so that the measure $d^{H^1} \gamma$ does not exist.

However, if one takes the formula (1.5) seriously for the moment, then one immediately obtains a "proof" of Thm. 1.1, by formally evaluating the integral with the method of stationary phase (often called Laplace's method in the case of a real phase): One "only" needs to pretend that $H_{xy}(M)$ is finite-dimensional, and use that at a minimal geodesic $\gamma \in \Gamma_{xy}^{\min}$, one has $E(\gamma) = d(x, y)^2/2$.

To prove Thm. 1.1 in this paper, we use finite-dimensional approximation to give a rigorous meaning to formula (1.5): Taking a partition $\tau = \{0 = \tau_0 < \tau_1 < \dots < \tau_N = 1\}$ of the time interval, we replace the infinite-dimensional manifold $H_{xy}(M)$ by the finite-dimensional submanifold $H_{xy;\tau}(M)$ of piecewise geodesics subordinated to the partition τ . These finite-dimensional integrals can then be evaluated using Laplace's method, and careful error estimates show that one can exchange taking the limit $t \rightarrow 0$ and the limit $|\tau| \rightarrow 0$. This will be explained in Section 3.

In physics, it is customary to regularize path integrals like (1.5) using zeta determinants as follows. A standard fact of Riemannian geometry is that the Hessian of the energy is given by

$$\nabla^2 E|_{\gamma}[X, Y] = (X, (-\nabla_s^2 + \mathcal{R}_{\gamma})Y)_{L^2}, \quad (1.6)$$

where $(\mathcal{R}_{\gamma}Y)(s) = R(\dot{\gamma}(s), Y(s))\dot{\gamma}(s)$ denotes the Jacobi endomorphism and

$$(X, Y)_{L^2} = \int_0^1 \langle X(s), Y(s) \rangle ds \quad (1.7)$$

is the L^2 metric along the paths. The observation is that the operator $-\nabla_s^2 + \mathcal{R}_{\gamma}$ as a Laplace type operator on the line with Dirichlet boundary conditions has a well-defined zeta determinant, and (for close points $x, y \in M$) one should have

$$\oint_{H_{xy}(M)} e^{-E(\gamma)/2t} [\gamma]_0^1{}^{-1} d^{H^1} \gamma \propto e^{-d(x,y)^2/4t} \det_{\zeta}(-\nabla_s^2 + \mathcal{R}_{\gamma})^{-1/2}$$

as $t \rightarrow 0$. Here we used the sign \propto , because a physicist's principle is that the zeta-renormalized value of path integral is merely well-defined "up to a multiplicative constant", which is only cancelled when one considers the quotient of two path integrals. Now this is exactly what we were doing in Thm. 1.1 (if we imagine both $p_t^L(x, y)$ and $e_t(x, y)$ being represented by a path integral as in (1.5)) and indeed, we obtain the following result:

Theorem 1.2. *Under the assumptions of Thm. 1.1, we have*

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{\Gamma_{xy}^{\min}} [\gamma]_0^1 \frac{\det_\zeta(-\nabla_s^2)^{1/2}}{\det'_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)^{1/2}} d^{L^2} \gamma. \quad (1.8)$$

Here we integrate with respect to the L^2 metric (1.7) on Γ_{xy}^{\min} and $\det'_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)$ denotes the zeta determinant of the Jacobi operator with the zero modes excluded.

Notice that in this case, we integrate over the path space using the L^2 metric so that no reference to the H^1 metric or path space is made. Hence there is an " H^1 picture" for the regularization of the path integral (1.5), where we choose the structure on the space of fields in just the right way to make the appearing determinants converge as honest infinite products of the eigenvalues, and an " L^2 picture", where we don't need to make such a choice but have to regularize the determinant using zeta function regularization.

One reason why results like Thm. 1.1 or Thm. 1.2 are interesting is that they give relations between geometric quantities on the path spaces of a Riemannian manifold and geometric quantities down on M . One such example is the Jacobian of the Riemannian exponential map,

$$J(x, y) = \det(d \exp_x |_{\dot{\gamma}_{xy}(0)}), \quad (1.9)$$

which is often called Van-Vleck-Morette-determinant in physics literature [PPV11, I.7]. $J(x, y)$ is a function on $M \bowtie M$, the set of points $(x, y) \in M \times M$ such that there exists a unique minimizing geodesic γ_{xy} joining the two. In formula (1.9), we take the differential of the Riemannian exponential map at the point $\dot{\gamma}_{xy}(0)$ to obtain a linear map $d \exp_x : T_{\dot{\gamma}_{xy}(0)} T_x M \cong T_x M \rightarrow T_y M$ and $J(x, y)$ is then the determinant of this linear map formed with help of the metric. It is well known (compare e.g. [BGV04, Section 2.5]) that in the case that $(x, y) \in M \bowtie M$, we have

$$\lim_{t \rightarrow 0} \frac{p_t^L(x, y)}{e_t(x, y)} = J(x, y)^{-1/2} [\gamma_{xy}]_0^1^{-1}. \quad (1.10)$$

Comparing this with the results of Thm. 1.1 and Thm. 1.2 gives the equalities

$$J(x, y) = \det(\nabla^2 E|_{\gamma_{xy}}) = \frac{\det_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)}{\det_\zeta(-\nabla_s^2)}. \quad (1.11)$$

While this may be useful to obtain information on $J(x, y)$ (for example, one directly sees that $J(x, y)$ is symmetric in x and y , which is not obvious from its definition), it seems

that the true power of this result lies in the reverse implications: Since $J(x, y)$ can be characterized as the solution to a certain ordinary differential equation (see (6.1), (6.2) below), we obtain a method to calculate infinite-dimensional determinants by solving an ordinary differential equation. This result is known as the Gel'fand-Yaglom theorem and we will prove it as an application in Section 6. Furthermore, using a degenerate Gel'fand-Yaglom theorem, we give a formula for the lowest order term in the heat kernel expansion using ODE along, without mentioning infinite determinants.

We hope that by comparing the higher order terms in the asymptotic expansion of $p_t^L(x, y)$ with the complete formal asymptotic expansion of the path integral (1.5), one obtains further interesting results like this. This will be done in a subsequent paper.

This paper is structured as follows. First, we review some theory on infinite-dimensional determinants in Hilbert spaces and we prove an approximation theorem on Fredholm determinants (Thm. 2.3) which seems to be new in this form, but is necessary for the proof of Thm. 1.1. In the next section, we show how to represent the heat kernel p_t^L by approximating the path integral (1.5) with finite-dimensional path integrals. Afterwards, we combine these techniques to give a proof of Thm. 1.1. In Section 5, we give a brief summary of the theory of zeta determinants and prove Thm 1.2. Afterwards, we give an application of our results by proving the Gel'fand Yaglom theorem, as mentioned above.

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2 Fredholm Determinants and the Hessian of the Energy

For $a, b \in \mathbb{R}$, $a < b$, write $I := [a, b]$. Let M be a Riemannian manifold of dimension n . For a smooth path γ in M parametrized by I , consider the operator $P := -\nabla_s^2$ on $L^2(I, \gamma^*TM)$ with Dirichlet boundary conditions. It is straightforward to show that P is essentially self-adjoint on the domain $C_c^\infty(I, \gamma^*TM)$ (the space of compactly supported sections of γ^*TM) and self-adjoint on the Sobolev space $H_0^2(I, \gamma^*TM)$. Its eigenvalues can be explicitly computed: For a parallel orthonormal frame $e_1(s), \dots, e_n(s)$ of TM along γ , the sections E_{ik} , $i = 1, \dots, n$, $k = 1, 2, \dots$, given by

$$E_{ik}(a + s) := \sqrt{\frac{2}{b-a}} \sin\left(\frac{\pi ks}{b-a}\right) e_i(a + s), \quad 0 \leq s \leq b-a \quad (2.1)$$

form an orthonormal basis of $L^2(I, \gamma^*TM)$ (the completeness can be easily checked using the Stone-Weierstraß theorem for locally compact spaces [dB59]). Obviously, the corresponding eigenvalues to E_{ik} are the numbers

$$\lambda_k := \frac{\pi^2 k^2}{(b-a)^2}, \quad k = 1, 2, \dots \quad (2.2)$$

each eigenvalue having multiplicity n .

Since the operator P is positive and self-adjoint, we can form the powers P^m for $m \in \mathbb{R}$ and define the Sobolev spaces

$$H_0^m(I, \gamma^*TM) := P^{-m/2}L^2(I, \gamma^*TM) \subset H^m(I, \gamma^*TM)$$

with the Sobolev norm

$$\|X\|_{H^m} := \|P^{m/2}X\|_{L^2}, \quad (2.3)$$

which is non-degenerate because P has a trivial kernel. By definition, this norm turns the map $P^{m/2} : H_0^l(I, \gamma^*TM) \rightarrow H_0^{l-m}(I, \gamma^*TM)$ into an isometry, for any $m, l \in \mathbb{R}$.

Notice that for smooth $X \in H_0^1(I, \gamma^*TM)$, we have

$$(P^{1/2}X, P^{1/2}X)_{L^2} = (PX, X)_{L^2} = -(\nabla_s^2 X, X)_{L^2} = (\nabla_s X, \nabla_s X)_{L^2} = \|X\|_{H^1}^2$$

so that for $m = 1$, the norm defined in (2.3) coincides with the H^1 norm defined before in (1.3) and there is no ambiguity in the notation. In particular, in the case that $I = [0, 1]$, we have

$$H_0^1(I, \gamma^*TM) \cong T_\gamma H_{xy}(M),$$

where $x := \gamma(0)$, $y := \gamma(1)$. Of course, orthonormal bases on the spaces $H_0^m(I, \gamma^*TM)$ can be obtained by rescaling the L^2 orthonormal basis (2.1) appropriately. In particular, the basis

$$F_{ik}(a+s) := \frac{\sqrt{2(b-a)}}{\pi k} \sin\left(\frac{\pi ks}{b-a}\right) e_i(a+s), \quad 0 \leq s \leq b-a, \quad (2.4)$$

$i = 1, \dots, n$, $k = 1, 2, \dots$, is an orthonormal basis of $H_0^1(I, \gamma^*TM)$.

For later use, we need the following two lemmas.

Lemma 2.1. *For any $m \in \mathbb{R}$, the inclusion of $H_0^{m+1}(I, \gamma^*TM)$ into $H_0^m(I, \gamma^*TM)$ is a Hilbert-Schmidt operator. Furthermore, the inclusion operator from $H_0^{m+2}(I, \gamma^*TM)$ into $H_0^m(I, \gamma^*TM)$ is nuclear, and P^{-1} is trace-class when considered as a bounded operator on $H_0^m(I, \gamma^*TM)$.*

Proof. Denote the inclusion operator from H_0^{m+1} into H_0^m by J_m . In the case $m = 1$, we have using the orthonormal basis (2.4) of $H_0^1(I, \gamma^*TM)$ that

$$\|J_0\|_2^2 = \sum_{i=1}^n \sum_{k=1}^\infty \|J_0 F_{ik}\|_{L^2}^2 = \sum_{i=1}^n \sum_{k=1}^\infty \|F_{ik}\|_{L^2}^2 = n \sum_{k=1}^\infty \frac{(b-a)^2}{\pi^2 k^2} = (b-a)^2 \frac{n}{6},$$

where we used that $\sum_{k=1}^\infty 1/k^2 = \pi^2/6$ [Eul40]. For $m \neq 1$, we have $J_m = P^{-m/2}J_0P^{m/2}$, so that J_m is also Hilbert-Schmidt by the ideal property of Hilbert-Schmidt operators.

The inclusion of $H_0^{m+2}(I, \gamma^*TM)$ into $H_0^m(I, \gamma^*TM)$ is equal to $J_m J_{m+1}$ and the composition of two Hilbert-Schmidt operators is trace-class, so the second statement follows.

Finally, we can write

$$[P^{-1} : H_0^m \rightarrow H_0^m] = J_m J_{m+1} [P^{-1} : H_0^m \rightarrow H_0^{m+2}],$$

which finishes the proof, because nuclear operators form an ideal. \square

Lemma 2.2. *For any $l, m \in \mathbb{R}$ with $l \leq m$, we have*

$$\|P^{(l-m)/2} X\|_{H^m} = \|X\|_{H^l} \leq \left(\frac{b-a}{\pi}\right)^{m-l} \|X\|_{H^m}.$$

Proof. Using the basis E_{ik} from (2.1) to the eigenvalues λ_k , decompose a given vector field $X \in H_0^m([a, b], \gamma^* TM)$ as $X = \sum_{i=1}^n \sum_{k=1}^\infty X_{ik} E_{ik}$. Then for any $l \leq m$, we have

$$\|X\|_{H^m}^2 = \sum_{i=1}^n \sum_{k=1}^\infty \lambda_k^m |X_{ik}|^2 \geq \lambda_1^{m-l} \sum_{i=1}^n \sum_{k=1}^\infty \lambda_k^l |X_{ik}|^2 = \left(\frac{\pi^2}{(b-a)^2}\right)^{m-l} \|X\|_{H^l}^2,$$

using the explicit value for λ_1 as in (2.2). This is the statement. \square

Let us now discuss the determinant of the Hessian of the energy. If T is a bounded linear operator on a separable Hilbert space \mathcal{H} , then its determinant can be defined if it has the form $T = \text{id} + W$ with a trace-class operator W . We will call such operators *determinant-class* and their (*Fredholm*) *determinant* can be defined by

$$\det(T) := \prod_{j=1}^\infty (1 + \lambda_j), \quad (2.5)$$

where λ_j are the eigenvalues of W , repeated with algebraic multiplicity. Because as a trace-class operator, W is compact, its non-zero spectrum consists only of eigenvalues of finite algebraic multiplicity (see e.g. Thm. 7.1 in [Con94]) and the trace-class condition means just that $\sum_{j=1}^\infty |\lambda_j| < \infty$, which implies that the product converges absolutely. In particular, $\det(T) = 0$ if and only if $\lambda_j = -1$ for some j , i.e. T has the eigenvalue zero. There are many other ways to define the determinant of T , see [Sim77]. For us, the following equivalent way to calculate a determinant will be useful.

Theorem 2.3. *Let \mathcal{H} be a separable Hilbert space and let $T := \text{id} + W$ be a bounded operator on T with W trace-class. Let $V_1 \subseteq V_2 \subseteq \dots$ be a nested sequence of finite-dimensional subspaces such that their union is dense in \mathcal{H} . Then we have*

$$\det(T) = \lim_{k \rightarrow \infty} \det(T|_{V_k}).$$

Remark 2.4. In particular, if e_1, e_2, \dots is an orthonormal basis of \mathcal{H} , then setting V_k to be the span of e_1, \dots, e_k yields that

$$\det(T) = \lim_{k \rightarrow \infty} \det\left(\langle e_i, T e_j \rangle\right)_{1 \leq i, j \leq k},$$

where the latter is an ordinary determinant of matrices.

Proof. Let π_k be the orthogonal projection on V_k . Because $\text{id} + \pi_k W \pi_k$ has the block diagonal form

$$\text{id} + \pi_k W \pi_k = \begin{pmatrix} T|_{V_k} & 0 \\ 0 & \text{id} \end{pmatrix}$$

with respect to the orthogonal splitting $\mathcal{H} = V_k \oplus V_k^\perp$, we have

$$\det(T|_{V_k}) = \det(\text{id} + \pi_k W \pi_k),$$

where the right hand side denotes the Fredholm determinant on \mathcal{H} . Let n_k be the dimension of V_k and let e_1, e_2, \dots be an orthonormal basis of \mathcal{H} such that e_1, \dots, e_{n_k} is an orthonormal basis of V_k . Using this orthonormal basis, we have

$$\text{tr } W_k = \sum_{j=1}^{\infty} \langle e_j, \pi_k W \pi_k e_j \rangle = \sum_{j=1}^{n_k} \langle e_j, W e_j \rangle \longrightarrow \sum_{j=1}^{\infty} \langle e_j, W e_j \rangle = \text{tr } W. \quad (2.6)$$

For the Hilbert-Schmidt norm, we find

$$\|W_k - W\|_2^2 = \sum_{i,j=1}^{\infty} \langle e_i, (\pi_k W \pi_k - W) e_j \rangle^2 = \sum_{\{i,j \mid i > n_k \text{ or } j > n_k\}} \langle e_i, W e_j \rangle^2,$$

which converges to zero since W is Hilbert-Schmidt (this follows e.g. from the dominated convergence theorem). Thus $W_k \rightarrow W$ in the Hilbert-Schmidt norm.

The 2-regularized determinant of a determinant-class operator $\text{id} + V$ is defined by

$$\det_2(\text{id} + V) = \det(\text{id} + V) e^{-\text{tr } V},$$

see Section 6 in [Sim77]. Because \det_2 is continuous in the topology induced by Hilbert-Schmidt norm (Thm. 6.5 in [Sim77]) and because of (2.6), we have

$$\lim_{k \rightarrow \infty} \det(\text{id} + W_k) = \lim_{k \rightarrow \infty} \det_2(\text{id} + W_k) e^{\text{tr } W_k} = \det_2(\text{id} + W) e^{\text{tr } W} = \det(\text{id} + W).$$

This finishes the proof. \square

For $s \in I$, define the *Jacobi endomorphism* by

$$\mathcal{R}_\gamma(s)v := R(\dot{\gamma}(s), v)\dot{\gamma}(s), \quad v \in T_{\gamma(s)}M, \quad (2.7)$$

where R is the Riemann curvature tensor of M . Because of the symmetries of R , \mathcal{R}_γ is a symmetric endomorphism field of the bundle γ^*TM over I . Since \mathcal{R}_γ is smooth and uniformly bounded on I , we can form the operator $P + \mathcal{R}_\gamma$, which is then self-adjoint on the same domain as P , and possesses the same mapping properties as P .

From now on, suppose that γ is a geodesic. Then the Hessian $\nabla^2 E|_\gamma$ is given by

$$\nabla^2 E|_\gamma[X, Y] = (\nabla_s X, \nabla_s Y)_{L^2} + (X, \mathcal{R}_\gamma Y)_{L^2} = (X, (P + \mathcal{R}_\gamma)Y)_{L^2} \quad (2.8)$$

for $X, Y \in H_0^1(I, \gamma^*TM)$, see e.g. Thm. 13.1 in [Mil63]. Hence on $H_0^1(I, \gamma^*TM) \subset L^2(I, \gamma^*TM)$, the Hessian is given by the operator $P + \mathcal{R}_\gamma$ with respect to the L^2 metric. Of course, this operator is far from being determinant-class, since it is even unbounded. But by (2.8), we also have

$$\nabla^2 E|_\gamma[X, Y] = (X, Y)_{H^1} + (P^{-1}\mathcal{R}_\gamma X, Y)_{H^1} = (X, P^{-1}(P + \mathcal{R}_\gamma)Y)_{H^1}, \quad (2.9)$$

so on the space $H_0^1(I, \gamma^*TM)$, the bilinear form $\nabla^2 E|_\gamma$ is given by the operator $\text{id} + P^{-1}\mathcal{R}_\gamma$. Now, indeed, $P^{-1}\mathcal{R}_\gamma$ is trace-class on $H_0^1(I, \gamma^*TM)$, by Lemma 2.1. In fact, we can even calculate its value in terms of a curvature integral, as the following proposition shows.

Proposition 2.5 (The Hessian of the Energy). *Let $\gamma \in H_{xy}(M)$ be a geodesic and consider $\nabla^2 E|_\gamma$ as an operator on $T_\gamma H_{xy}(M)$, by dualizing with the H^1 metric. Then $\nabla^2 E|_\gamma - \text{id}$ is trace-class with*

$$\text{Tr}(\nabla^2 E|_\gamma - \text{id}) = - \int_0^1 \text{ric}(\dot{\gamma}(s), \dot{\gamma}(s)) s(1-s) ds,$$

where ric denotes the Ricci tensor of M .

Remark 2.6. This implies that $\nabla^2 E|_\gamma$ is determinant-class as a bilinear form on the Hilbert space $H_0^1([0, 1], \gamma^*TM)$ with respect to the H^1 metric. Furthermore, it is easy to see from the above considerations that $\nabla^2 E|_\gamma$ is determinant-class on $H_0^m([0, 1], \gamma^*TM)$ if and only if $m = 1$.

Proof. By (2.9), we have using the orthonormal basis F_{ik} from (2.4) that

$$\begin{aligned} \text{Tr}(\nabla^2 E|_\gamma - \text{id}) &= \sum_{i=1}^n \sum_{k=1}^\infty (P^{-1}\mathcal{R}_\gamma F_{ik}, F_{ik})_{H^1} = \sum_{i=1}^n \sum_{k=1}^\infty (\mathcal{R}_\gamma F_{ik}, F_{ik})_{L^2} \\ &= \int_0^1 \underbrace{\left(\sum_{j=1}^n \langle R(\dot{\gamma}(s), e_i(s)) \dot{\gamma}(s), e_i(s) \rangle \right)}_{= -\text{ric}(\dot{\gamma}(s), \dot{\gamma}(s))} \left(\sum_{k=1}^\infty \frac{2}{\pi^2 k^2} \sin(\pi s k)^2 \right) ds. \end{aligned}$$

Now because of $2 \sin(z)^2 = 1 - \cos(2z)$, we have

$$\sum_{k=1}^\infty \frac{2}{\pi^2 k^2} \sin(\pi s k)^2 = \frac{1}{\pi^2} \sum_{k=1}^\infty \frac{1}{k^2} - \sum_{k=1}^\infty \frac{1}{\pi^2 k^2} \cos(2\pi s k) = s(1-s),$$

where we used the Fourier transform identity of the second Bernoulli polynomial [Sch13],

$$\sum_{k=1}^\infty \frac{1}{\pi^2 k^2} \cos(2\pi k s) = s^2 - s + \frac{1}{6}. \quad \square$$

Example 2.7 (Constant Curvature Manifolds). We calculate $\det(\nabla^2 E|_\gamma)$ in the case that $\gamma \in \Gamma_{xy}^{\min} \subset H_{xy}(M)$ for a Riemannian manifold M of constant sectional curvature κ . In this special case, the Jacobi eigenvalue equation along a geodesic γ is (see e.g. [Cha84, p. 63])

$$(P + \mathcal{R}_\gamma(s))X(s) = -\nabla_s^2 X(s) - \kappa|\dot{\gamma}(s)|^2 X(s) + \kappa \langle X(s), \dot{\gamma}(s) \rangle \dot{\gamma}(s) = \lambda X(s).$$

Because γ is a geodesic, the eigenspaces separate into spaces of vector fields that are either parallel to $\dot{\gamma}$ or orthogonal to $\dot{\gamma}$. Write $r := |\dot{\gamma}(s)| = d(x, y)$ (which is independent of s because γ is a geodesic). Set $e_1(s) := \dot{\gamma}(s)/r$ and let $e_2(s), \dots, e_n(s)$ be a parallel orthonormal basis of the orthogonal complement of $\dot{\gamma}$ along γ .

If we use the frame $e_1(s), \dots, e_n(s)$ to define the orthonormal basis F_{ik} as in (2.4), then this is an orthonormal basis of eigenvectors of $P + \mathcal{R}_\gamma$ on the space $H_0^1([0, 1], \gamma^* TM)$: The F_{1k} are eigenvectors to the eigenvalues $\lambda_k = \pi^2 k^2$ (so these have multiplicity one each), while the F_{ik} , $i = 2, \dots, n$, are eigenvectors to the eigenvalues $\mu_k = \pi^2 k^2 - \kappa r^2$ (each of these has multiplicity $n - 1$). The eigenvalues for the operator $\text{id} + P^{-1}\mathcal{R}_\gamma$ are then

$$\tilde{\lambda}_k = \frac{\lambda_k}{\pi^2 k^2} = 1, \quad \tilde{\mu}_k = \frac{\mu_k}{\pi^2 k^2} = 1 - \frac{\kappa r^2}{\pi^2 k^2}. \quad (2.10)$$

(If $\kappa > 0$, this reflects that in order to have no zero eigenvalues, we need to have $r^2 \kappa < \pi^2$.) We obtain by (2.5) and (2.9)

$$\det(\nabla^2 E|_\gamma) = \det(\text{id} + P^{-1}\mathcal{R}_\gamma) = \prod_{k=1}^{\infty} \left(1 - \frac{\kappa r^2}{\pi^2 k^2}\right)^{n-1} = \left(\frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}r}\right)^{n-1}$$

by the product formula for the sine [FB05, p. 220] (if κ is negative, then \sin becomes \sinh). Note that these results coincide with the explicit formulas for the Jacobian of the exponential map $J(x, y)$ on manifolds with constant curvature [Hsu02, Example 5.1.2]. This is no coincidence, as we will see in Corollary 4.7 below.

3 The Heat Kernel as a Path Integral

Throughout, let M be a compact Riemannian manifold of dimension n . To give a rigorous meaning to the path integral appearing in (1.5), we use finite-dimensional approximation. For a partition $\tau = \{0 = \tau_0 < \tau_1 < \dots < \tau_N = 1\}$ of the interval $[0, 1]$, write

$$H_{xy;\tau}(M) := \{\gamma \in H_{xy}(M) \mid \gamma|_{[\tau_{j-1}, \tau_j]} \text{ is a unique minimizing geodesic}\}, \quad (3.1)$$

for the space of paths γ where each segment $\gamma|_{[\tau_{j-1}, \tau_j]}$ is a minimizing geodesic between its endpoints and the endpoints are not in each other's cut locus. One can show that $H_{xy;\tau}(M)$ is an $n(N - 1)$ -dimensional submanifold of $H_{xy}(M)$ and that the map

$$\text{ev}_\tau : H_{xy;\tau}(M) \longrightarrow M^{N-1}, \quad \gamma \longmapsto (\gamma(\tau_1), \dots, \gamma(\tau_{N-1})) \quad (3.2)$$

is an open embedding. The H^1 metric (1.3) turns $H_{xy;\tau}(M)$ into a finite-dimensional Riemannian manifold.

Let now M be compact and let L be a self-adjoint Laplace type operator, acting on sections of a metric vector bundle \mathcal{V} over M . Any such operator can uniquely be written as $L = \nabla^* \nabla + V$, where ∇ is a metric connection on \mathcal{V} and V is a symmetric endomorphism field. L generates a strongly continuous semigroup of operators e^{-tL} , the heat semigroup, where for each $t > 0$, the operator e^{-tL} has a smooth integral kernel $p_t^L(x, y)$, the heat kernel, which is a section of the bundle $\mathcal{V} \boxtimes \mathcal{V}^*$ over $M \times M$, which has fibers $\mathcal{V}_x \otimes \mathcal{V}_y^* \cong \text{Hom}(\mathcal{V}_y, \mathcal{V}_x)$ over points $(x, y) \in M \times M$. It has been shown by the author [Lud16a] that one has

$$p_t^L(x, y) = \lim_{|\tau| \rightarrow 0} (4\pi t)^{-n/2} \oint_{H_{xy;\tau}(M)} e^{-E(\gamma)/2t} \mathcal{P}(\gamma)^{-1} d^{\Sigma-H^1} \gamma, \quad (3.3)$$

where the limit goes over any sequence of partitions the mesh of which tends to zero, the integral over $H_{xy;\tau}(M)$ is taken with respect to a discretized version of the H^1 volume and the slash over the integral sign denotes division by $(4\pi t)^{\dim(H_{xy;\tau}(M))/2}$. Formula (3.3) also involves the *path-ordered exponential* $\mathcal{P}(\gamma)$ determined by L which is obtained by solving a certain ODE along the paths γ and which reduces to the usual parallel transport in the case that the potential V vanishes. For previous results on path integrals by finite-dimensional approximation on manifolds, see for example [AD99], [Lim07], [BP08], [Bär12] or [Lae13].

Unfortunately, the result (3.3) cannot be used to perform the proof of Thm. 1.1 envisioned in the introduction, because we have no control over the error of the path integral approximation: There is no reason why one should obtain the same result when taking the limit $|\tau| \rightarrow 0$ first and then $t \rightarrow 0$ as opposed to vice versa, especially if one divides by $e_t(x, y)$ previously.

To fix this, we add certain correction terms to the integrand of (3.3), which improve the time-uniformity of the approximation. In order to do this, we use that $p_t^L(x, y)$ has an asymptotic expansion near the diagonal of the form

$$p_t^L(x, y) \sim e_t(x, y) \sum_{j=0}^{\infty} t^j \frac{\Phi_j(x, y)}{j!},$$

where the Φ_j are certain smooth sections of $\mathcal{V} \boxtimes \mathcal{V}^*$ near the diagonal, determined by transport equations, compare e.g. [BGV04, Section 2] or [Lud16b, Thm. 1.1]. It is well known that $\Phi_0(x, y) = [\gamma_{xy} \|_0^1]^{-1} J(x, y)^{-1/2}$, where $[\gamma_{xy} \|_0^1]$ denotes parallel transport along the unique shortest geodesic γ_{xy} between x and y and $J(x, y)$ is the Jacobian of the exponential map defined in (1.9) (see [BGV04, Section 2.5]). In particular, for $(x, y) \in M \bowtie M$, we have

$$\lim_{t \rightarrow 0} \frac{p_t^L(x, y)}{e_t(x, y)} = \Phi_0(x, y) = [\gamma_{xy} \|_0^1]^{-1} J(x, y)^{-1/2}. \quad (3.4)$$

By the results of [Lud16b], the heat kernel $p_t^L(x, y)$ can be approximated by the path

integral

$$I(t, x, y) := (4\pi t)^{-nN/2} \int_{H_{xy;\tau}(M)} e^{-E(\gamma)/2t} \Upsilon^\tau(\gamma) d^{H^1} \gamma, \quad (3.5)$$

where $\Upsilon^\tau(\gamma)$ is given by

$$\Upsilon^\tau(\gamma) = |\det(\text{dev}_\tau|_\gamma)| \prod_{j=1}^N \chi(d(\gamma(\tau_{j-1}), \gamma(\tau_j))) \frac{\Phi_0(\gamma(\tau_{j-1}), \gamma(\tau_j))}{(\Delta_j \tau)^{n/2}} \quad (3.6)$$

formed from the determinant of the evaluation map (3.2), the first heat kernel coefficient Φ_0 and a smooth cutoff function $\chi : [0, \infty) \rightarrow \mathbb{R}$ with $\chi(r) = 1$ near zero and $\chi(r) = 0$ for $r \geq \text{inj}(M)$, the injectivity radius of M . That $p_t^L(x, y)$ can be approximated by the integral $I(t, x, y)$ precisely means the following.

Proposition 3.1. *There exist constants $\delta, C > 0$ such that*

$$|p_t^L(x, y) - I(t, x, y)| \leq C t p_t^\Delta(x, y)$$

for all partitions τ of the interval $[0, 1]$ with $|\tau| \leq \delta$.

This is a special case of Thm. 1.2 combined with Lemma 5.7 in [Lud16b]. Relevant for us is the following corollary.

Corollary 3.2. *Under the assumptions of Thm. 1.1, we have*

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{\Gamma_{xy}^{\min}} \frac{\Upsilon^\tau(\gamma)}{\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma.$$

for any partition τ with $|\tau| \leq \delta$, where $\delta > 0$ is as in Prop. 3.1. Here by $\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})$, we mean the determinant of $\nabla^2 E$ when restricted to the normal space of Γ_{xy}^{\min} inside $H_{xy;\tau}(M)$.

Proof. By Prop. 3.1 and the calculations above, there exist constants $\delta > 0$ and $C_1 > 0$ such that

$$\left| \frac{p_t^L(x, y)}{e_t(x, y)} - \frac{I(t, x, y)}{e_t(x, y)} \right| \leq C_1 \frac{p_t^\Delta(x, y)}{e_t(x, y)} t \quad (3.7)$$

for all $x, y \in M$, all partitions τ with $|\tau| \leq \delta$ and each $0 < t \leq 1$. By Thm. 5.2 in [Lud16b], we have

$$\frac{p_t^\Delta(x, y)}{e_t(x, y)} \leq C_2 t^{-k/2} \quad (3.8)$$

for some constant $C_2 > 0$. Using this on (3.7), multiplying by $(4\pi t)^{k/2}$ and taking the limit $t \rightarrow 0$, we get, using the definition of $I(t, x, y)$,

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \lim_{t \rightarrow 0} (4\pi t)^{-n(N-1)/2+k/2} \int_{H_{xy;\tau}(M)} e^{-[E(\gamma)-d(x,y)^2/2]/2t} \Upsilon^\tau(\gamma) d^{H^1} \gamma.$$

The limit on the right hand side can now be evaluated using Laplace's method if Γ_{xy}^{\min} is a k -dimensional non-degenerate submanifold of $H_{xy}(M)$. Namely then, Γ_{xy}^{\min} is also a non-degenerate submanifold of $H_{xy;\tau}(M)$ for each partition τ fine enough. The function $\phi(\gamma) := E(\gamma) - d(x, y)^2/2$ is zero on the submanifold Γ_{xy}^{\min} and positive everywhere else. Therefore, by Laplace's method (compare e.g. Appendix A in [Lud16b]), the integral therefore concentrates on Γ_{xy}^{\min} in the limit $t \rightarrow 0$. The precise result is

$$\lim_{t \rightarrow 0} (4\pi t)^{-n(N-1)/2+k/2} \int_{H_{xy;\tau}(M)} e^{-\phi(\gamma)/2t} \Upsilon^\tau(\gamma) d^{H^1} \gamma = \int_{\Gamma_{xy}^{\min}} \frac{\Upsilon^\tau(\gamma)}{\det_\tau(\nabla^2 \phi|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma,$$

where the determinant of $\nabla^2 \phi$ is taken over the (finite-dimensional) normal space of Γ_{xy}^{\min} inside $H_{xy;\tau}(M)$. Clearly, $\nabla^2 \phi = \nabla^2 E$ so the result follows. \square

4 Heat Kernel Asymptotics as a Fredholm Determinant

In this section, we prove Thm. 1.1. Before starting the proof, let us shed some light on the assumptions of the theorem.

Lemma 4.1. *Let M be a complete manifold and suppose that for $x, y \in M$, the set Γ_{xy}^{\min} of length minimizing geodesics is a submanifold of $H_{xy}(M)$. Then the following statements are equivalent.*

- (i) Γ_{xy}^{\min} is non-degenerate in the sense that for each $\gamma \in \Gamma_{xy}^{\min}$, the Hessian of the energy $\nabla^2 E$ is non-degenerate when restricted to the normal bundle $N_\gamma \Gamma_{xy}^{\min}$ of Γ_{xy}^{\min} at γ .
- (ii) For each $\gamma \in \Gamma_{xy}^{\min}$ and each Jacobi field X along γ with $X(0) = X(1) = 0$, there exists a geodesic variation γ_ε , $\varepsilon \in (-\epsilon_0, \epsilon_0)$ with $\frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} \gamma_\varepsilon = X$ and $\gamma_\varepsilon(0) = x$, $\gamma_\varepsilon(1) = y$ for each $\varepsilon \in (-\epsilon_0, \epsilon_0)$.

Notice that if $X \in T_\gamma \Gamma_{xy}^{\min}$, we always have $\nabla^2 E[X, Y] = 0$ for all $Y \in T_\gamma H_{xy}(M)$, so that unless $\dim(\Gamma_{xy}^{\min}) = 0$, $\nabla^2 E$ is always degenerate on $T_\gamma H_{xy}(M)$.

Proof. Let $\gamma \in \Gamma_{xy}^{\min}$ and suppose there exists $X \in N_\gamma \Gamma_{xy}^{\min}$ such that $\nabla^2 E|_\gamma[X, Y] = 0$ for all $Y \in T_\gamma H_{xy}(M)$. By (2.8), X is a weak solution of the equation $(-\nabla_s^2 + \mathcal{R}_\gamma)X = 0$, therefore smooth by elliptic regularity and hence a strong solution. That is, X is a Jacobi field with $X(0) = X(1) = 0$. However, there exists no geodesic variation γ_ε in $H_{xy}(M)$ with $\frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} \gamma_\varepsilon = X$, because then we would have $\gamma_\varepsilon \in \Gamma_{xy}^{\min}$ for each ε and $\frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} \gamma_\varepsilon \in T\Gamma_{xy}^{\min}$, a contradiction to $X \in N_\gamma \Gamma_{xy}^{\min}$.

Conversely, let X be a Jacobi field along $\gamma \in \Gamma_{xy}^{\min}$ with $X(0) = X(1) = 0$. If there exists no geodesic variation $\gamma_\varepsilon \in \Gamma_{xy}^{\min}$ with $\frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} \gamma_\varepsilon = X$, then this means that $X \notin T_\gamma \Gamma_{xy}^{\min}$. Hence its normal component \tilde{X} is not zero, and \tilde{X} is a Jacobi field, because all elements of $T_\gamma \Gamma_{xy}^{\min}$ are Jacobi fields, and \tilde{X} is the difference of such a tangent vector and the Jacobi field X . This implies that $\nabla^2 E|_\gamma[\tilde{X}, Y] = 0$ for all $Y \in T_\gamma H_{xy}(M)$, so that $\nabla^2 E|_\gamma$ is degenerate, even when restricted to the normal space $N_\gamma \Gamma_{xy}^{\min}$. \square

Remark 4.2. Versions of Thm. 1.1 can also be obtained in the case that Γ_{xy}^{\min} is a *degenerate* submanifold of $H_{xy}(M)$, compare e.g. [Mol75, Section 3]. We restrict to the non-degenerate case for simplicity.

Remark 4.3. It is easy to see that a version of Thm. 1.1 also holds in the case that Γ_{xy}^{\min} is a disjoint union of a k -dimensional submanifold Γ_0 and submanifolds $\Gamma_1, \dots, \Gamma_\ell$ of lower dimension, under the assumption that all of them are all non-degenerate. In that case, (1.4) still holds if one replaces the integral over Γ_{xy}^{\min} by an integral over Γ_0 .

Let us now go on with the proof of Thm. 1.1. Remember that so far (Corollary 3.2), we have shown that

$$\lim_{|\tau| \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{\Gamma_{xy}^{\min}} \frac{\Upsilon^\tau(\gamma)}{\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma.$$

for any partition τ small enough, where $\Upsilon^\tau(\gamma)$ is given by (3.6) and $\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})$ is the finite-dimensional determinant of $\nabla^2 E$, restricted to the normal space of Γ_{xy}^{\min} inside $H_{xy;\tau}(M)$. It remains to show that the integrand under the integral over Γ_{xy}^{\min} can be replaced by the integrand of Thm. 1.1. To obtain Thm. 1.1 from this, we need the following three technical lemmas, the (somewhat involved) proofs of which will be given at the end of this section.

Lemma 4.4. *There exists a constant $C > 0$ such that for each $\gamma \in \Gamma_{xy}^{\min}$, we have*

$$\left| \prod_{j=1}^N \Phi_0(\gamma(\tau_{j-1}), \gamma(\tau_j)) - [\gamma]_0^1 \right| \leq C|\tau|$$

for each partition τ of the interval $[0, 1]$ fine enough.

Lemma 4.5. *Let M be a compact Riemannian manifold and $x, y \in M$. Then for every $C > 0$, there exist constants $\alpha > 0$ and $N_0 \in \mathbb{N}$ such that the following holds: For any geodesic $\gamma \in \Gamma_{xy}^{\min}$ in M , we have*

$$e^{-\alpha|\tau|^{-3}} \leq |\det(\text{dev}_\tau|_\gamma)| \prod_{j=1}^N (\Delta_j \tau)^{-n/2} \leq e^{\alpha|\tau|^{-3}}$$

for any partition $\tau = \{0 = \tau_0 < \tau_1 < \dots < \tau_N = 1\}$ of the interval $[0, 1]$ with $N \geq N_0$ and $|\tau| \leq C/N$.

Lemma 4.6. *Let S be a set of partitions of the interval $[0, 1]$ such that for any $\varepsilon > 0$, there exists $\tau \in S$ with $|\tau| < \varepsilon$. Then for any $\gamma \in \Gamma_{xy}^{\min}$, the union of the spaces $T_\gamma H_{xy;\tau}(M)$, $\tau \in S$ is dense in $T_\gamma H_{xy}(M) = H_0^1([0, 1], \gamma^* TM)$.*

Using these Lemmas, we can now prove our main result.

Proof (of Thm. 1.1). By Corollary 3.2, we are done if we show that

$$\lim_{|\tau| \rightarrow 0} \int_{\Gamma_{xy}^{\min}} \frac{\Upsilon^\tau(\gamma)}{\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma = \int_{\Gamma_{xy}^{\min}} \frac{[\gamma]_0^1^{-1}}{\det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma.$$

where we may take the limit over any suitable sequence $(\tau^{(k)})_{k \in \mathbb{N}}$ of partitions satisfying $|\tau^{(k)}| \rightarrow 0$ (notice that a particular result of the lemma is that the value of the integral on the left hand side integral does not depend on τ).

By Lemma 4.4 and Lemma 4.5, the function $\Upsilon^\tau(\gamma)$ is uniformly bounded on Γ_{xy}^{\min} and we have for any $\gamma \in \Gamma_{xy}^{\min}$ that

$$\lim_{|\tau| \rightarrow 0} \Upsilon^\tau(\gamma) = [\gamma]_0^1^{-1},$$

where for a fixed $C > 0$, the limit goes over any sequence $\tau^{(k)}$ of partitions with $|\tau^{(k)}| \rightarrow 0$, where each such τ additionally satisfies $|\tau^{(k)}| \leq C/N$.

By Lemma 4.6, for any such sequence $(\tau^{(k)})_{k \in \mathbb{N}}$, the union of the spaces $T_\gamma H_{xy; \tau^{(k)}}(M)$, $k \in \mathbb{N}$, is dense in $T_\gamma H_{xy}(M)$ for every $\gamma \in \Gamma_{xy}^{\min}$. Furthermore, also the union of the spaces $N_\gamma \Gamma_{xy}^{\min} \cap T_\gamma H_{xy; \tau^{(k)}}(M)$ is dense in $N_\gamma \Gamma_{xy}^{\min}$. (For let $X \in N_\gamma \Gamma_{xy}^{\min}$. Then there exists a sequence $X_k \in T_\gamma H_{xy; \tau^{(k)}}(M)$ with $\|X - X_k\|_{H^1} \rightarrow 0$ by Lemma 4.6. But if $Y_k \in T_\gamma \Gamma_{xy}^{\min}$ is the part of X_k tangent to Γ_{xy}^{\min} , we have

$$\|X - X_k\|_{H^1}^2 = \|X - (X_k - Y_k)\|_{H^1}^2 + \|Y_k\|_{H^k}^2,$$

so that $X_k - Y_k$ is an approximating sequence of X in $N_\gamma \Gamma_{xy}^{\min} \cap T_\gamma H_{xy; \tau^{(k)}}(M)$.) By Thm. 2.3, we therefore have

$$\lim_{k \rightarrow \infty} \det_{\tau^{(k)}}(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}) = \lim_{k \rightarrow \infty} \det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min} \cap T_\gamma H_{xy; \tau^{(k)}}(M)}) = \det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})$$

if we additionally choose the sequence $(\tau^{(k)})_{k \in \mathbb{N}}$ to be nested (since then the corresponding sequence of spaces $N_\gamma \Gamma_{xy}^{\min} \cap T_\gamma H_{xy; \tau^{(k)}}(M)$ is nested, too).

We obtain that if for a fixed $C > 0$, we take the limit over some nested sequence of partitions $\tau^{(k)}$ with $|\tau^{(k)}| \rightarrow 0$ that additionally satisfies $|\tau^{(k)}| \leq C/N$, then the integrand from Corollary 3.2 converges to the integrand from the theorem pointwise.

To justify the exchange of integration and taking the limit, we give a uniform bound on $\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})$. Because of (2.9), we have

$$\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}) := \det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min} \cap H_{xy; \tau}(M)}) = \det((\text{id} + \pi_\tau P^{-1} \mathcal{R}_\gamma \pi_\tau)|_{N_\gamma \Gamma_{xy}^{\min}}),$$

where π_τ is the orthogonal projection of $H_{xy}(M)$ onto $H_{xy; \tau}(M)$. Because of the standard estimate for Fredholm determinants (see [Sim77, Thm. 3.2])

$$e^{-\|T\|_1} \leq \det(\text{id} + T) \leq e^{\|T\|_1}, \quad (4.1)$$

which holds for all trace-class operators T , we have

$$\det_\tau(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{-1/2} \leq e^{\|\pi_\tau P^{-1} \mathcal{R}_\gamma \pi_\tau\|_1/2}.$$

But

$$\|\pi_\tau P^{-1} \mathcal{R}_\gamma \pi_\tau\|_1 \leq \|\pi_\tau\| \|P^{-1} \mathcal{R}_\gamma\|_1 \|\pi_\tau\| \leq \|P^{-1}\|_1 \|\mathcal{R}_\gamma\|,$$

which is finite by Lemma 2.1 and bounded uniformly over $\gamma \in \Gamma_{xy}^{\min}$ since Γ_{xy}^{\min} is compact. The proof now follows from Lebesgue's theorem of dominated convergence. \square

Restricting to the case $(x, y) \in M \bowtie M$ gives the following corollary.

Corollary 4.7 (The Jacobian of the Exponential Map). *Let M be a complete Riemannian manifold. Let $(x, y) \in M \bowtie M$ and let γ_{xy} be the unique minimizing geodesic connecting x to y in time one. Then we have*

$$\det(\nabla^2 E|_{\gamma_{xy}}) = J(x, y),$$

where $J(x, y)$ is the Jacobian determinant of the exponential map, as defined in (1.9). Here, $H_{xy}(M)$ carries the H^1 metric (1.3).

Proof. Of course, this is a local result, so in the case that M is non-compact, we can take some patch of M containing γ_{xy} and embed it isometrically into some compact Riemannian manifold M' in such a way that γ_{xy} is still a minimizing geodesic, without changing $J(x, y)$ or the determinant of the Hessian of the energy. This shows that we may assume that M is compact so that the above results apply.

Taking the heat kernel of the Laplace-Beltrami operator in Thm. 1.1 and restricting to the case $(x, y) \in M \bowtie M$ (which implies $\Gamma_{xy}^{\min} = \{\gamma_{xy}\}$ and $k = \dim(\Gamma_{xy}^{\min}) = 0$), we have

$$J(x, y)^{-1/2} = \Phi_0(x, y) = \lim_{t \rightarrow 0} \frac{p_t^\Delta(x, y)}{e_t(x, y)} = \det(\nabla^2 E|_{\gamma_{xy}})^{-1/2}.$$

by (3.5). \square

Example 4.8 (The first Coefficient on Spheres). On an n -dimensional sphere S_R^n of radius $R = 1/\sqrt{\kappa}$, the determinant of the Hessian of the energy, respectively the Jacobian of the exponential map, is given by

$$J(x, y) = \left(\frac{\sin(\sqrt{\kappa} d(x, y))}{\sqrt{\kappa} d(x, y)} \right)^{n-1}, \quad (4.2)$$

in the case that x and y are not antipodal points, see [Hsu02, Example 5.1.2]. From this, one can read off the heat kernel asymptotics of Thm. 1.1 in this case (compare (3.4) and Corollary 4.7). We now use Thm. 1.1 to calculate the first coefficient for the Laplace-Beltrami operator on S_R^n in the case that x and y are antipodal points.

Without loss of generality, we assume that $x = (R, 0, \dots, 0)$ and $y = (-R, 0, \dots, 0)$ are the north and south pole. In this case, the set Γ_{xy}^{\min} is diffeomorphic to S_R^{n-1} , the $n-1$ -dimensional sphere of radius R , via the diffeomorphism

$$\rho : S_R^{n-1} \longrightarrow \Gamma_{xy}^{\min} \quad \theta \longmapsto \left[s \mapsto \begin{pmatrix} R \cos(\pi s) \\ \theta \sin(\pi s) \end{pmatrix} \right].$$

For $v \in T_\theta S_R^{n-1}$, we have

$$d\rho|_\theta v =: X_v = \left[s \mapsto \begin{pmatrix} 0 \\ v \sin(\pi s) \end{pmatrix} \right].$$

Since $v \in T_\theta S_R^{n-1}$, we have $\langle v, \theta \rangle = 0$, hence $\langle \dot{X}_v(s), \rho(\theta)(s) \rangle = 0$ so that

$$\nabla_s X_v(s) = \dot{X}_v(s) - \kappa \langle \dot{X}_v(s), \rho(\theta)(s) \rangle \rho(\theta)(s) = -\pi \begin{pmatrix} 0 \\ v \sin(\pi s) \end{pmatrix},$$

by the explicit formula for the Levi-Civita connection on the round sphere. Therefore, if e_1, \dots, e_{n-1} is an orthonormal basis of $T_\theta S_R^{n-1}$, the Jacobian determinant of ρ is given by

$$\begin{aligned} |\det(d\rho|_\theta)| &= \det\left((X_{e_i}, X_{e_j})_{H^1}\right)_{1 \leq i, j \leq n-1}^{1/2} = \det\left(\pi^2 \langle e_i, e_j \rangle \int_0^1 \cos(\pi s)^2 ds\right)_{1 \leq i, j \leq n-1}^{1/2} \\ &= \pi^{n-1} 2^{(1-n)/2}, \end{aligned}$$

which is constant. To calculate the determinant of the Hessian of the energy, remember that the eigenvalues are given by (2.10). In our case, $r = R\pi$ and $\kappa = 1/R^2$ so $\tilde{\mu}_1 = 0$, which has to be left out to calculate the Hessian on the normal space to Γ_{xy}^{\min} . We obtain

$$\det(\nabla^2 E|_{N_{\Gamma_{xy}^{\min}}}) = \prod_{k=2}^{\infty} \tilde{\mu}_k^{n-1} = \prod_{k=2}^{\infty} \left(1 - \frac{\kappa r^2}{\pi^2 k^2}\right)^{n-1} = \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right)^{n-1} = 2^{1-n},$$

because the product "telescopes", that is

$$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \lim_{N \rightarrow \infty} \left(\prod_{k=2}^N \frac{k-1}{k}\right) \left(\prod_{k=2}^N \frac{k+1}{k}\right) = \lim_{N \rightarrow \infty} \frac{1}{N} \frac{N+1}{2} = \frac{1}{2}.$$

Therefore, by Thm. 1.1, we have

$$\begin{aligned} \lim_{t \rightarrow 0} (4\pi t)^{(1-n)/2} \frac{p_t^L(x, y)}{e_t(x, y)} &= \int_{\Gamma_{xy}^{\min}} 2^{(n-1)/2} d^{H^1} \gamma = 2^{(n-1)/2} \int_{S_R^{n-1}} \det(d\rho|_\theta) d\theta \\ &= \pi^{n-1} R^{n-1} \text{vol}(S^{n-1}) = 2 \frac{\pi^{3n/2-1} R^{n-1}}{\Gamma(n/2)}. \end{aligned}$$

This result can also be found in [Hsu02, Example 5.3.3].

To finish this section, it is left to prove the Lemmas 4.4, 4.5 and 4.6.

Proof (of Lemma 4.4). By (3.4), we have

$$\prod_{j=1}^N \Phi_0(\gamma(\tau_{j-1}), \gamma(\tau_j)) - [\gamma]_0^1 = \left(\prod_{j=1}^N J(\gamma(\tau_{j-1}), \gamma(\tau_j))^{-1/2} - 1 \right) [\gamma]_0^1. \quad (4.3)$$

By Corollary II.8.1 in [Cha06] and compactness of M , there exist constants $C_1, R > 0$ such that $|J(x, y) - 1| \leq C_1 d(x, y)^2$ for all $x, y \in M$ with $d(x, y) < R$. Hence for each $\alpha \in \mathbb{R}$, there exists a constant C_α such that

$$J(x, y)^\alpha \leq e^{C_\alpha d(x, y)^2}.$$

for such $x, y \in M$. Because $d(\gamma(\tau_{j-1}), \gamma(\tau_j)) = \Delta_j \tau d(x, y)$, we have

$$\prod_{j=1}^N J(\gamma(\tau_{j-1}), \gamma(\tau_j))^\alpha \leq e^{C_\alpha d(x, y)^2 \sum_{j=1}^N (\Delta_j \tau)^2} \leq e^{C_\alpha |\tau| d(x, y)^2}$$

Using this for $\alpha = \pm 1/2$ gives the lemma together with (4.3). \square

Proof (of Lemma 4.5). Identify the tangent spaces $T_{\gamma(s)}M$ with $T_{\gamma(0)}M$ using parallel transport along γ . Let $\tau = \{0 = \tau_0 < \tau_1 < \dots < \tau_N = 1\}$, $N \geq 2$, be a partition of the interval $[0, 1]$ and write for abbreviation $\Delta_j := \Delta_j \tau = \tau_j - \tau_{j-1}$ throughout this proof.

Step 1. Define the subspace $W_\tau \subset T_\gamma H_{xy}(M) = H_0^1([0, 1], \gamma^* TM)$ by

$$W_\tau := \{X \in T_\gamma H_{xy}(M) \mid X \text{ smooth on } (\tau_{j-1}, \tau_j) \text{ with } \nabla_s^2 X(s) = 0\}. \quad (4.4)$$

This means that elements $X \in W_\tau$ are piecewise linear, i.e. they have the form

$$X(\tau_{j-1} + s) = \left(1 - \frac{s}{\Delta_j}\right) v_{j-1} + \frac{s}{\Delta_j} v_j, \quad v_j := X(\tau_j), \quad 0 \leq s \leq \Delta_j. \quad (4.5)$$

Define

$$\Psi_\tau : \bigoplus_{j=1}^N T_{\gamma(\tau_j)} M \longrightarrow W_\tau, \quad (v_1, \dots, v_{N-1}) \longmapsto X_v,$$

where X_v is the unique element in W_τ with $X_v(\tau_j) = v_j$ (where we set $v_0 = v_N = 0$). Then by the explicit form (4.5) of $X_v = \Psi_\tau(v_1, \dots, v_{N-1})$, $X_w = \Psi_\tau(w_1, \dots, w_{N-1})$, we have (using the convention $v_0 = v_N = w_0 = w_N = 0$)

$$\begin{aligned} (X_v, X_w)_{H^1} &= \sum_{j=1}^N \int_{\tau_{j-1}}^{\tau_j} \left\langle \frac{1}{\Delta_j} (v_j - v_{j-1}), \frac{1}{\Delta_j} (w_j - w_{j-1}) \right\rangle ds \\ &= \sum_{j=1}^N \frac{1}{\Delta_j} (\langle v_j, w_j \rangle + \langle v_{j-1}, w_{j-1} \rangle - \langle v_j, w_{j-1} \rangle - \langle v_{j-1}, w_j \rangle) \\ &= \left\langle \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}, D_\tau \begin{pmatrix} w_1 \\ \vdots \\ w_{N-1} \end{pmatrix} \right\rangle \end{aligned}$$

where D_τ is the $n(N-1) \times n(N-1)$ matrix

$$D_\tau := \begin{pmatrix} \left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}\right) \text{id} & -\frac{1}{\Delta_2} \text{id} & & & \\ -\frac{1}{\Delta_2} \text{id} & \left(\frac{1}{\Delta_2} + \frac{1}{\Delta_3}\right) \text{id} & \ddots & & \\ & \ddots & \ddots & -\frac{1}{\Delta_{N-1}} \text{id} & \\ & & & -\frac{1}{\Delta_{N-1}} \text{id} & \left(\frac{1}{\Delta_{N-1}} + \frac{1}{\Delta_N}\right) \text{id} \end{pmatrix}.$$

Using induction and Laplace's formula for determinants, one shows that $\det(D_\tau) = \prod_{j=1}^N \Delta_j^{-n}$. As a subspace of $H_0^1([0, 1], \gamma^* TM)$, W_τ carries the induced H^1 scalar product. With respect to this scalar product, we obtain that

$$|\det(\Psi_\tau)| = \det(\Psi_\tau^* \Psi_\tau)^{1/2} = \det(D_\tau)^{1/2} = \prod_{j=1}^N \Delta_j^{-n/2}. \quad (4.6)$$

Step 2. Define the operator

$$K_\tau : W_\tau \longrightarrow T_\gamma H_{xy}(M), \quad X \longmapsto K_\tau X, \quad (4.7)$$

where $Y := K_\tau X$ is the unique solution of

$$\begin{cases} -\nabla_s^2 Y(s) + \mathcal{R}_\gamma(s)Y(s) = -\mathcal{R}_\gamma(s)X(s) & \text{for } s \neq \tau_j \\ Y(\tau_j) = 0 & \text{for } j = 1, \dots, N, \end{cases}$$

with \mathcal{R}_γ the curvature endomorphism along γ considered in Section 2. This problem indeed has a unique solution, because $Y = K_\tau X$ is just patched together from the unique solutions of Dirichlet problems on each subinterval $[\tau_{j-1}, \tau_j]$. Namely, the self-adjoint operator $-\nabla_s^2 + \mathcal{R}_\gamma$ with Dirichlet boundary conditions is invertible on each of the subintervals $[\tau_{j-1}, \tau_j]$, because it has trivial kernel: Elements in the kernel are Jacobi fields with vanishing endpoints. A non-zero element in the kernel would therefore imply that $\gamma(\tau_{j-1})$ and $\gamma(\tau_j)$ are conjugate, which cannot happen for $N \geq 2$ as γ is a minimizing geodesic. Because the right hand side is smooth on these subintervals, Y is as well. For $X \in W_\tau$, set $\tilde{X} := X + K_\tau X := X + Y$. Then $\tilde{X} \in T_\gamma H_{xy;\tau}(M)$, because for $s \neq \tau_j$, we have

$$\nabla_s^2 \tilde{X} = \underbrace{\nabla_s^2 X(s)}_{=0} + \nabla_s^2 Y(s) = \mathcal{R}_\gamma(s)Y(s) + \mathcal{R}_\gamma(s)X(s) = \mathcal{R}_\gamma(s)\tilde{X}(s).$$

Thus \tilde{X} is a piecewise Jacobi field, i.e. an element of $T_\gamma H_{xy;\tau}(M)$. Notice that

$$\text{id} + K_\tau : W_\tau \longrightarrow T_\gamma H_{xy;\tau}(M)$$

is an isomorphism of vector spaces, because the dimensions coincide and it is injective: If $X = -K_\tau X$, for $X \in W_\tau$, then in particular $X(\tau_j) = -(K_\tau X)(\tau_j) = 0$ for all j , hence $X = 0$. Furthermore, for vectors $v_j \in T_{\gamma(\tau_j)} M$, $X := (\text{id} + K_\tau)\Psi_\tau(v_1, \dots, v_{N-1})$ is the piece-wise Jacobi field with $X(\tau_j) = v_j$. Therefore,

$$(\text{dev}_\tau|_\gamma)^{-1} = (\text{id} + K_\tau)\Psi_\tau. \quad (4.8)$$

Extend K_τ to a bounded linear operator on $T_\gamma H_{xy}(M)$ through extension by zero on the orthogonal complement W_τ^\perp . Denote by i_τ, p_τ and ι_τ, π_τ the inclusions and orthogonal projections corresponding to the subspaces W_τ respectively $T_\gamma H_{xy;\tau}(M)$ of $T_\gamma H_{xy}(M)$. Using (4.8) and (4.6), we obtain

$$|\det(\text{dev}_\tau|_\gamma)| \prod_{j=1}^N \Delta_j^{-n/2} = |\det(\pi_\tau(\text{id} + K_\tau)i_\tau)|^{-1} \frac{\prod_{j=1}^N \Delta_j^{-n/2}}{|\det(\Psi_\tau)|} = |\det(\pi_\tau(\text{id} + K_\tau)i_\tau)|^{-1}$$

Furthermore,

$$\begin{aligned} |\det(\pi_\tau(\text{id} + K_\tau)i_\tau)| &= \det(p_\tau(\text{id} + K_\tau)^* \iota_\tau \pi_\tau(\text{id} + K_\tau)i_\tau)^{1/2} \\ &= \det(p_\tau(\text{id} + K_\tau)^*(\text{id} + K_\tau)i_\tau)^{1/2}, \end{aligned}$$

where in the last step we used that the image of $\text{id} + K_\tau$ is contained in $T_\gamma H_{xy;\tau}(M)$ so that the projection and inclusion in the middle can be left out. For $X_1, X_2 \in W_\tau$, let $Y_1 := K_\tau X_1, Y_2 := K_\tau X_2$ and calculate

$$(X_1, K_\tau X_2)_{H^1} = (X_1, Y_2)_{H^1} = \sum_{j=1}^N \int_{\tau_{j-1}}^{\tau_j} \langle \nabla_s X_1(s), \nabla_s Y_2(s) \rangle ds = 0,$$

which follows from integration by parts since $\nabla_s^2 X_1 = 0$ for $s \in [\tau_{j-1}, \tau_j]$ and $Y_2(\tau_j) = Y_2(\tau_{j-1}) = 0$ for all $j = 1, \dots, N$. This shows $K_\tau X \subset W_\tau^\perp$. Put together, we get for $X_1, X_2 \in W_\tau$ that

$$\begin{aligned} (X_1, (\text{id} + K_\tau)^*(\text{id} + K_\tau)X_2)_{H^1} &= (X_1, X_2)_{H^1} + \underbrace{(X_1, K_\tau X_2)_{H^1}}_{=0} + \underbrace{(K_\tau X_1, X_2)_{H^1}}_{=0} + (K_\tau X_1, K_\tau X_2)_{H^1} \\ &= (X_1, (\text{id} + K_\tau^* K_\tau)X_2)_{H^1}, \end{aligned}$$

i.e. $p_\tau(\text{id} + K_\tau)^*(\text{id} + K_\tau)i_\tau = p_\tau(\text{id} + K_\tau^* K_\tau)i_\tau$, and

$$\det(p_\tau(\text{id} + K_\tau)^*(\text{id} + K_\tau)i_\tau)^{1/2} = \det(p_\tau(\text{id} + K_\tau^* K_\tau)i_\tau)^{1/2} = \det(\text{id} + K_\tau^* K_\tau)^{1/2}, \quad (4.9)$$

where the last determinant is a Fredholm determinant and the last step uses that $\text{id} + K_\tau^* K_\tau$ has block diagonal form with respect to the decomposition $T_\gamma H_{xy}(M) = W_\tau \oplus W_\tau^\perp$.

Therefore, with a view on the standard determinant estimate (4.1), we are led to estimate $\|K_\tau^* K_\tau\|_1 = \text{tr}(K_\tau^* K_\tau) = \|K_\tau\|_2^2$, the Hilbert-Schmidt norm of K_τ .

Step 3. We need some preliminary considerations. Let $[a, b]$ be any subinterval of $[0, 1]$ and write P for the operator $-\nabla_s^2$ on $L^2([a, b], \gamma^* TM)$ with Dirichlet boundary conditions, as in Section 2. Suppose that $[a, b] \subsetneq [0, 1]$. Then $P + \mathcal{R}_\gamma$ is an isomorphism from $H_0^m([a, b], \gamma^* TM)$ to $H_0^{m-2}([a, b], \gamma^* TM)$ for each $m \in \mathbb{R}$ (remember that γ is a minimizing geodesic, hence $\gamma|_{[a, b]}$ is unique minimizing, so there are no non-trivial Jacobi fields with vanishing endpoints along $\gamma|_{[a, b]}$, i.e. the kernel of $P + \mathcal{R}_\gamma$ is trivial). We now show that

$$\|(P + \mathcal{R}_\gamma)^{-1}X\|_{H^1} \leq \frac{(b-a)^2}{\pi^2 - \|\mathcal{R}_\gamma\|(b-a)^2} \|X\|_{H^1}, \quad (4.10)$$

for each $X \in H^1([a, b], \gamma^*TM)$ and any $\gamma \in \Gamma_{xy}^{\min}$, where $\|\mathcal{R}_\gamma\|$ is the operator norm of the operator $X \mapsto \mathcal{R}_\gamma X$ on $H_0^1([0, 1], \gamma^*TM)$. First we have using Lemma 2.2 above that

$$\|P^{-1}\mathcal{R}_\gamma X\|_{H^1} \leq \frac{(b-a)^2}{\pi^2} \|\mathcal{R}_\gamma X\|_{H^1} \leq \frac{(b-a)^2}{\pi^2} \|\mathcal{R}_\gamma\| \|X\|_{H^1},$$

since the operator norm of \mathcal{R}_γ on $[a, b]$ is less or equal to the operator norm of \mathcal{R}_γ on the interval $[0, 1]$. We find for all $X \in H_0^1([a, b], \gamma^*TM)$ that

$$\|(\text{id} + P^{-1}\mathcal{R}_\gamma)X\|_{H^1} \geq \|X\|_{H^1} - \|P^{-1}\mathcal{R}_\gamma X\|_{H^1} \geq \left(1 - \|\mathcal{R}_\gamma\| \frac{(b-a)^2}{\pi^2}\right) \|X\|_{H^1}.$$

Because $\text{id} + P^{-1}\mathcal{R}_\gamma$ is self-adjoint on $H_0^1([a, b], \gamma^*TM)$ as is easy to verify, we obtain for its smallest eigenvalue

$$\mu_{\min} = \inf_{X \neq 0} \frac{\|(\text{id} + P^{-1}\mathcal{R}_\gamma)X\|_{H^1}}{\|X\|_{H^1}} \geq \left(1 - \|\mathcal{R}_\gamma\| \frac{(b-a)^2}{\pi^2}\right).$$

The spectral radius of the inverse $(\text{id} + P^{-1}\mathcal{R}_\gamma)^{-1}$ is equal to $1/\mu_{\min}$. Since $\text{id} + P^{-1}\mathcal{R}_\gamma$ is self-adjoint on $H_0^1([a, b], \gamma^*TM)$ and so is its inverse, the spectral radius equals the operator norm, whence

$$\|(\text{id} + P^{-1}\mathcal{R}_\gamma)^{-1}X\|_{H^1} \leq \frac{1}{\mu_{\min}} \|X\|_{L^2} \leq \frac{\pi^2}{\pi^2 - \|\mathcal{R}_\gamma\|(b-a)^2} \|X\|_{H^1}$$

Finally, using Lemma 2.2 again, we get

$$\begin{aligned} \|(P + \mathcal{R}_\gamma)^{-1}X\|_{H^1} &= \|P^{-1}(\text{id} + P^{-1}\mathcal{R}_\gamma)^{-1}X\|_{H^1} \\ &\leq \frac{(b-a)^2}{\pi^2} \|(\text{id} + P^{-1}\mathcal{R}_\gamma)^{-1}X\|_{H^1} \\ &\leq \frac{(b-a)^2}{\pi^2 - \|\mathcal{R}_\gamma\|(b-a)^2} \|X\|_{H^1}, \end{aligned}$$

which is the claim.

Step 4. We finally derive a bound on $\|K_\tau\|_2^2$. For any vector $X \in T_\gamma H_{xy;\tau}(M)$ and any $j = 1, \dots, N$, we have $K_\tau X|_{[\tau_{j-1}, \tau_j]} = -(P + \mathcal{R}_\gamma)^{-1}\mathcal{R}_\gamma X|_{[\tau_{j-1}, \tau_j]}$, where $(P + \mathcal{R}_\gamma)^{-1}$ is the operator discussed in Step 3 on the interval $[a, b] := [\tau_{j-1}, \tau_j]$.

Let $E_1, E_2, \dots, E_{n(N-1)}$ be an orthonormal basis of W_τ . Using the estimate (4.10) from

Step 3 on the operator norm of $(P + \mathcal{R}_\gamma)^{-1}$ on $H^1([\tau_{j-1}, \tau_j], \gamma^*TM)$, we obtain

$$\begin{aligned}
\|K_\tau\|_2^2 &= \sum_{i=1}^{n(N-1)} \|K_\tau E_i\|_{H^1}^2 = \sum_{i=1}^{n(N-1)} \sum_{j=1}^N \|K_\tau E_i|_{[\tau_{j-1}, \tau_j]}\|_{H^1}^2 \\
&= \sum_{i=1}^{n(N-1)} \sum_{j=1}^N \|-(P + \mathcal{R}_\gamma)^{-1} \mathcal{R}_\gamma E_i|_{[\tau_{j-1}, \tau_j]}\|_{H^1}^2 \\
&\leq \sum_{i=1}^{n(N-1)} \sum_{j=1}^N \left(\frac{\Delta_j^2}{\pi^2 - \|\mathcal{R}_\gamma\| \Delta_j^2} \right)^2 \|\mathcal{R}_\gamma E_i|_{[\tau_{j-1}, \tau_j]}\|_{H^1}^2 \\
&\leq \sum_{i=1}^{n(N-1)} \left(\frac{|\tau|^2}{\pi^2 - \|\mathcal{R}_\gamma\| |\tau|^2} \right)^2 \|\mathcal{R}_\gamma E_i\|_{H^1}^2 \leq n(N-1) \left(\frac{\|\mathcal{R}_\gamma\| |\tau|^2}{\pi^2 - \|\mathcal{R}_\gamma\| |\tau|^2} \right)^2
\end{aligned}$$

We now suppose that $|\tau| \leq C/N$ for some $C > 0$. Suppose additionally the partition τ be so fine that $|\tau| \leq \pi/\sqrt{2\|\mathcal{R}_\gamma\|}$, or equivalently $\pi^2 - \|\mathcal{R}_\gamma\| |\tau|^2 \geq \pi^2/2$. By the assumption $|\tau| \leq C/N$, this is the case in particular if $N \geq N_0 := \lceil C\sqrt{2\|\mathcal{R}_\gamma\|/\pi} \rceil$. For such τ , we have

$$\frac{\|\mathcal{R}_\gamma\| |\tau|^2}{\pi^2 - \|\mathcal{R}_\gamma\| |\tau|^2} \leq \frac{2\|\mathcal{R}_\gamma\| |\tau|^2}{\pi^2} \leq \frac{2\|\mathcal{R}_\gamma\| C^2}{\pi^2 N^2} = \frac{N_0^2}{N^2}$$

and

$$\|K_\tau\|_2^2 \leq n(N-1) \left(\frac{N_0^2}{N^2} \right)^2 \leq nN_0^2 \frac{1}{N^3}.$$

With a view on (4.9), this concludes the proof using (4.1), because the operator norm $\|\mathcal{R}_\gamma\|$ is uniformly bounded for $\gamma \in \Gamma_{xy}^{\min}$. \square

Remark 4.9. Notice that if M is flat, we have $W_\tau = H_{xy;\tau}(M)$ and the operator K_τ of the above proof is zero. Hence in the flat case, we have

$$|\det(\text{dev}_\tau|_\gamma)| \prod_{j=1}^N (\Delta_j \tau)^{-n/2} \equiv 1,$$

for each partition τ .

Proof (of Lemma 4.6). The proof is divided into two steps.

Step 1. We first show that the union of the spaces W_τ for $\tau \in S$ is dense in the space $H_0^1([0, 1], \gamma^*TM)$, where W_τ is the space defined in (4.4). Namely, we claim that given a partition $\tau = \{0 = \tau_0 < \tau_1 < \dots < \tau_N = 1\}$, a vector field $X \in H_0^1([0, 1], \gamma^*TM)$ is in the orthogonal complement of W_τ if and only if $X(\tau_j) = 0$ for all $j = 1, \dots, N-1$. Indeed, for a given $v \in T_{\gamma(\tau_j)}M$, define $Y \in W_\tau$ by

$$Y(s) = \begin{cases} s(1 - \tau_j)v & s \leq \tau_j \\ (1 - s)\tau_j v & s \geq \tau_j, \end{cases}$$

where we identified the spaces $T_{\gamma(s)}M$ by parallel transport along γ . Then integrating by parts and using that $\nabla_s^2 X \equiv 0$ on (τ_{j-1}, τ_j) yields

$$(X, Y)_{H^1} = \sum_{j=1}^N \int_{\tau_{j-1}}^{\tau_j} \langle \nabla_s X(s), \nabla_s Y(s) \rangle ds = \sum_{j=1}^{N-1} \langle X(\tau_j), Y(\tau_j-) - Y(\tau_j+) \rangle = \langle X(\tau_j), v \rangle.$$

This proves the claim, since this scalar product is zero for all v chosen this way if and only if $X(\tau_j) = 0$ for all j .

Now suppose that $X \in H_{xy;\tau}(M)$ is in the orthogonal complement of W_τ for all $\tau \in S$. Then by the observation before, we obtain that necessarily $X(s) = 0$ for all $s \in [0, 1]$ for which there exists a partition $\tau \in S$ with $s \in \tau$. Because of the condition on the set S , the set of such s is dense in $[0, 1]$, so from continuity follows $X \equiv 0$. Therefore the union of all W_τ , $\tau \in S$ must be a dense subset of $H_{xy}(M)$.

Step 2. Suppose that $W_\tau \neq H_{xy;\tau}(M)$, i.e. $\mathcal{R}_\gamma \neq 0$ (otherwise, we are already done with the proof). Let $Y \in W_\tau$. Then if K_τ is the operator defined in (4.7), then $Y + K_\tau Y \in T_\gamma H_{xy;\tau}(M)$, as seen in Step 2 of the proof of Lemma 4.5 above. By (4.10), we have

$$\begin{aligned} \|K_\tau Y\|_{H^1}^2 &= \sum_{j=1}^N \left\| -(P + \mathcal{R}_\gamma)^{-1} \mathcal{R}_\gamma Y|_{\tau_{j-1}, \tau_j} \right\|_{H^1}^2 \leq \sum_{j=1}^N \left(\frac{\Delta_j^2}{\pi^2 - \|\mathcal{R}_\gamma\| \Delta_j^2} \right)^2 \|\mathcal{R}_\gamma Y|_{\tau_{j-1}, \tau_j}\|_{H^1}^2 \\ &\leq |\tau|^4 \frac{4}{\pi^4} \|\mathcal{R}_\gamma Y\|_{H^1}^2 \leq |\tau|^4 \frac{4}{\pi^4} \|\mathcal{R}_\gamma\|^2 \|Y\|_{H^1}^2 \end{aligned}$$

whenever $\pi^2 - \|\mathcal{R}_\gamma\| |\tau|^2 \leq \pi^2/2$, or equivalently $|\tau| \leq \pi/\sqrt{2\|\mathcal{R}_\gamma\|}$ (here $\|\mathcal{R}_\gamma\|$ is the operator norm of the operator $X \mapsto \mathcal{R}_\gamma X$ on $H_0^1([0, 1], \gamma^* TM)$). We conclude that the operator norm of the operators K_τ for $|\tau|$ small enough satisfies the bound $\|K_\tau\| \leq C|\tau|^2$ with a constant $C > 0$ independent of τ . Hence

$$\begin{aligned} \|X - (Y + K_\tau Y)\|_{H^1} &\leq \|X - Y\|_{H^1} + \|K_\tau Y\|_{H^1} \leq \|X - Y\|_{H^1} + \|K_\tau\| \|Y\|_{H^1} \\ &\leq \|X - Y\|_{H^1} + C|\tau|^2 (\|X - Y\|_{H^1} + \|X\|_{H^1}). \end{aligned}$$

Now given $\varepsilon > 0$, choose $\delta > 0$ such that

$$\delta^2 < \min \left\{ \frac{\varepsilon}{C(\varepsilon + 2\|X\|_{H^1})}, \frac{\pi^2}{2\|\mathcal{R}_\gamma\|} \right\}$$

and let $S' \subset S$ be the set containing all partitions $\tau \in S$ with $|\tau| \leq \delta$. Then S' still has the property from the lemma, so by Step 1, for some $\tau \in S'$, we find $Y \in W_\tau$ such that $\|X - Y\|_{H^1} < \varepsilon/2$. Then by the choice of δ , if $|\tau| \leq \delta$, we have $\|X - (Y + K_\tau Y)\|_{H^1} \leq \varepsilon$. Because ε was arbitrary and $Y + K_\tau Y \in H_{xy;\tau}(M)$, $\tau \in S$, this shows that the union of all $H_{xy;\tau}(M)$, $\tau \in S$ is dense in $H_0^1([0, 1], \gamma^* TM)$. \square

5 Zeta Determinants and the Gelfand-Yaglom Theorem

So far, we have seen that in the case that the set Γ_{xy}^{\min} of minimizing geodesics between the points x, y is a k -dimensional non-degenerate submanifold of $H_{xy}(M)$ (with respect to the energy functional), we have

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{\Gamma_{xy}^{\min}} \frac{[\gamma]_0^{-1}}{\det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma.$$

The expression on the right hand side depends on the choice of a Riemannian metric on the manifold $H_{xy}(M)$ in two ways: First, because we integrate over the submanifold Γ_{xy}^{\min} using the Riemannian volume density of the induced metric. Secondly, because we take the determinant of the bilinear form $\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}$ using the metric on $N_\gamma \Gamma_{xy}^{\min}$ (because to calculate the determinant of a bilinear form, we need a metric). In both cases, the H^1 metric (1.3) turned out to be the correct choice.

However, there is another possible choice for the determinant of an operator on an infinite-dimensional space: the zeta determinant, which is defined for a certain class of unbounded operators on a Hilbert space. This approach is often used in physics to assign finite values to otherwise ill-defined path integrals, see e.g. [Haw77] or [Wit99]. Because we have $\nabla^2 E|_\gamma[X, Y] = (X, (-\nabla_s^2 + \mathcal{R}_\gamma)Y)_{L^2}$ (see (2.8)), one could get the idea to replace the determinant of $\nabla^2 E|_\gamma$ by the zeta determinant of the Jacobi-operator $-\nabla_s^2 + \mathcal{R}_\gamma$. This determinant does not depend on the choice of a Sobolev metric on the path spaces. Instead, it only depends on the eigenvalues of $-\nabla_s^2 + \mathcal{R}_\gamma$, considered as an unbounded operator on the Hilbert space $L^2([0, 1], \gamma^* TM)$. Since the H^1 metric on $H_{xy}(M)$ does no longer play a role then, it seems that one should also equip Γ_{xy}^{\min} with another metric when performing the integral. Here the L^2 metric comes into play.

For an elliptic non-negative self-adjoint pseudo-differential operator P of order $d > 0$, acting on an m -dimensional compact manifold Σ , the *zeta function* ζ_P is defined by

$$\zeta_P(z) := \sum_{\lambda \neq 0} \lambda^{-z}, \quad (5.1)$$

where the sum runs over all non-zero eigenvalues λ of P . Here, Σ may have a boundary, in which case we need to give appropriate boundary conditions to P . This sum converges for $\operatorname{Re}(z) > m/d$; however, one can check that ζ_P possesses a meromorphic extension to all of \mathbb{C} and that zero is not a pole [Gil95, Section 1.12]. Therefore, one can define the *zeta-regularized determinant*

$$\det_\zeta(P) := e^{-\zeta_P'(0)}.$$

If P actually has zero eigenvalues that were left out in the sum (5.1), it is conventional to write $\det'_\zeta(P)$ instead. The definition is motivated by the fact that if one (formally!)

plugs the series (5.1) into the right hand side of this definition (which is not possible since one cannot evaluate it at zero), one obtains

$$e^{-\zeta'_P(0)} \stackrel{\text{formally}}{=} \prod_{\lambda \neq 0} \lambda,$$

the product of the non-zero eigenvalues, which of course diverges; the zeta determinant "magically" assigns a finite value to this product.

Example 5.1 (Dirichlet-Laplacian along a Geodesic). Let γ be a smooth path in an n -dimensional Riemannian manifold M parametrized by $[0, t]$. Already in Section 2, we found the eigenvalues of the operator $P = -\nabla_s^2$ with Dirichlet boundary conditions on the space $L^2([0, t], \gamma^*TM)$ to be the numbers $\lambda_k = \pi^2 k^2 / t^2$, each of multiplicity n . Hence for $\text{Re } z > 1/2$, we have

$$\zeta_P(z) = n \sum_{k=1}^{\infty} \left(\frac{\pi^2 k^2}{t^2} \right)^{-z} = n \frac{t^{2z}}{\pi^{2z}} \sum_{k=1}^{\infty} k^{-2z} = n \frac{t^{2z}}{\pi^{2z}} \zeta(2z),$$

where ζ without subscript denotes the usual Riemann zeta function. Therefore,

$$\zeta'_P(0) = 2n(\log(t) - \log(\pi))\zeta(0) + 2n\zeta'(0) = -n \log(2t)$$

as it is well known that $\zeta(0) = -1/2$ and $\zeta'(0) = -\log(2\pi)/2$ [Son94]. We obtain

$$\det_{\zeta}(-\nabla_s^2) = e^{-\zeta'_P(0)} = (2t)^n \tag{5.2}$$

for the zeta determinant. For the operators P^m , $m > 0$, one easily sees that $\zeta_{P^m}(z) = \zeta_P(mz)$, hence $\det_{\zeta}(P^m) = \det_{\zeta}(P)^m$.

More generally, the zeta determinant can be defined for a wide class of (necessarily unbounded) closed operators with discrete spectrum on an abstract Hilbert space H , called *zeta-admissible* (for the definition, see [Sco02, Section 2]). That an operator is zeta-admissible essentially means that it has a well-defined zeta function which does not have a pole at zero. We will not need the exact definition here (which is somewhat involved); we will only need that Laplace type operators P on intervals with Dirichlet boundary conditions are zeta-admissible, as well as their positive powers. Such operators P are well-known to be zeta-admissible; this can be shown e.g. using the heat trace expansion as in [Gil95, Section 10].

The following result then generates many more examples.

Proposition 5.2 (Multiplicativity). [Sco02, Thm. 2.18] *Let \mathcal{H} be a Hilbert space, let P be a closed and invertible operator on \mathcal{H} with positive spectrum and let $T := \text{id} + W$ with W trace-class on \mathcal{H} . If P is zeta-admissible, then so are PT and TP and we have*

$$\det_{\zeta}(PT) = \det_{\zeta}(TP) = \det_{\zeta}(P) \det(T),$$

where $\det(T)$ denotes the usual Fredholm determinant.

Remark 5.3. We generally have $\det_\zeta(AB) \neq \det_\zeta(A) \det_\zeta(B)$. Instead, the above product rule holds.

Corollary 5.4 (Zeta Relativity). *Let P_1, P_2 be positive self-adjoint Laplace type operators with Dirichlet boundary conditions on the interval $[0, t]$, acting on the bundle γ^*TM , where γ is a smooth path in some Riemannian manifold M . Suppose that the difference $P_1 - P_2$ is of order zero and that P_1 and P_2 have trivial kernels. Then $P_1^{-1}P_2$ is well defined and determinant-class on $L^2([0, t], \gamma^*TM)$ and we have*

$$\det(P_1^{-1}P_2) = \frac{\det_\zeta(P_2)}{\det_\zeta(P_1)},$$

where the left hand side is the usual Fredholm determinant.

Proof. Because P_1 has trivial kernel, its inverse P_1^{-1} is well defined by spectral calculus, and $P_1^{-1} : L^2([0, t], \gamma^*TM) \rightarrow H_0^2([0, t], \gamma^*TM)$ is a bounded operator. By Lemma 2.1, the inclusion $H_0^2([0, t], \gamma^*TM) \rightarrow L^2([0, t], \gamma^*TM)$ is nuclear; hence the operator $P_1^{-1} : L^2([0, t], \gamma^*TM) \rightarrow L^2([0, t], \gamma^*TM)$ is trace-class, because it can be written as the composition of a bounded operator and a nuclear operator.

Write $P_2 = P_1 + V$ for an endomorphism field $V \in C^\infty([0, t], \gamma^*TM)$. Then

$$P_1^{-1}P_2 = P_1^{-1}(P_1 + V) = \text{id} + P_1^{-1}V$$

is determinant-class, because $P_1^{-1}V$ is trace-class. We can now apply Prop. 5.2 on the Hilbert space $L^2([0, t], \mathbb{R}^n)$ with $P = P_1$ and $T = P_1^{-1}P_2$ to obtain the required determinant identity. \square

Similarly, the following is true.

Proposition 5.5. *Let M be a Riemannian manifold and let $(x, y) \in M \bowtie M$. Then we have*

$$\det(\nabla^2 E|_{\gamma_{xy}}) = \frac{\det_\zeta(-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}})}{\det_\zeta(-\nabla_s^2)} = 2^{-n} \det_\zeta(-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}}),$$

where γ_{xy} is the unique minimizing geodesic travelling from x to y in time one and $-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}}$ is the Jacobi operator as in Section 2. Both operators on the right hand side carry Dirichlet boundary conditions.

Combining this with Corollary 4.7 and Example 5.1, we may express the Jacobian of the exponential map as the zeta determinant of the Jacobi operator.

Corollary 5.6. *Let M be a Riemannian manifold and $(x, y) \in M \bowtie M$. Then for any $t > 0$,*

$$J(x, y) = \frac{\det_\zeta(-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}})}{\det_\zeta(-\nabla_s^2)},$$

where $J(x, y)$ denotes the Jacobian of the exponential map, as in Remark (1.9).

Proof (of Prop. 5.5). Write $P := -\nabla_s^2$ and $\gamma := \gamma_{xy}$ for abbreviation. By (2.9), we have

$$\nabla^2 E|_\gamma[X, Y] = (X, P^{-1}(P + \mathcal{R}_\gamma)Y)_{H^1}.$$

Set $T := P^{-1}(P + \mathcal{R}_\gamma)$. Because $P^{-1/2} : L^2([0, t], \gamma^*TM) \rightarrow H_0^1([0, t], \gamma^*TM)$ is an isometry, we have

$$\det(\nabla^2 E|_\gamma) = \det^{H^1}(T) = \det^{L^2}(P^{1/2}TP^{-1/2}) = \det^{L^2}(P^{-1/2}(P + \mathcal{R}_\gamma)P^{-1/2}).$$

The operator $P^{-1/2}(P + \mathcal{R}_\gamma)P^{-1/2}$ is indeed determinant-class, since

$$P^{-1/2}(P + \mathcal{R}_\gamma)P^{-1/2} = \text{id} + P^{-1/2}\mathcal{R}_\gamma P^{-1/2} =: \text{id} + \tilde{W},$$

where \tilde{W} is the composition of two Hilbert-Schmidt operators and a bounded operator, hence trace-class. Set $W := P^{-1}\mathcal{R}_\gamma$. Then by Prop. 5.2,

$$\begin{aligned} \det^{L^2}(\text{id} + \tilde{W}) \det_\zeta(P^{1/2}) &= \det_\zeta((\text{id} + \tilde{W})P^{1/2}) = \det_\zeta(P^{1/2}(\text{id} + W)) \\ &= \det_\zeta(P^{1/2}) \det^{L^2}(\text{id} + W), \end{aligned}$$

since $P^{1/2}$ is zeta-admissible. This shows that the L^2 -determinant of $\text{id} + \tilde{W}$ is equal to the L^2 -determinant of $\text{id} + W = P^{-1}(P + \mathcal{R}_\gamma)$ (the latter now being an operator on $L^2([0, 1], \gamma^*TM)$). The result now follows from Corollary 5.4. \square

In the particular case $(x, y) \in M \bowtie M$, we obtain

$$\lim_{t \rightarrow 0} \frac{p_t^L(x, y)}{e_t(x, y)} = \frac{\det'_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)^{-1/2}}{\det_\zeta(-\nabla_s^2)^{-1/2}} [\gamma \|_0^1]^{-1}.$$

Now we prove the general case, Thm. 1.2.

Proof (of Thm. 1.2). By Thm. 1.1, we have

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{\Gamma_{xy}^{\min}} \frac{[\gamma \|_0^1]^{-1}}{\det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2}} d^{H^1} \gamma,$$

when Γ_{xy}^{\min} is endowed with the H^1 metric (1.3). By the transformation formula, we have

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{\Gamma_{xy}^{\min}} \frac{[\gamma \|_0^1]^{-1}}{\det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2} \det(\text{id}|_\gamma)} d^{L^2} \gamma, \quad (5.3)$$

where $\det(\text{id}|_\gamma)$ denotes the determinant of the identity map from Γ_{xy}^{\min} with the H^1 metric to the same space with the L^2 metric. Fix $\gamma \in \Gamma_{xy}^{\min}$ and let f_1, \dots, f_k be an H^1 -orthonormal basis of $T_\gamma \Gamma_{xy}^{\min} \cong \ker(P + \mathcal{R}_\gamma)$. Then

$$\det(\text{id}|_\gamma) = \det\left((f_i, f_j)_{L^2}\right)_{1 \leq i, j \leq k}^{1/2}. \quad (5.4)$$

Notice that f_1, \dots, f_k are smooth by elliptic regularity. Let f_{k+1}, f_{k+2}, \dots be a smooth H^1 -orthonormal basis of $N_\gamma \Gamma_{xy}^{\min}$. By Thm. 2.3 (respectively Remark 2.4) and (2.8), we have

$$\det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}) = \lim_{N \rightarrow \infty} \det \left((f_i, (P + \mathcal{R}_\gamma) f_j)_{L^2} \right)_{k+1 \leq i, j \leq N}. \quad (5.5)$$

Let Π be the H^1 -orthogonal projection in $H_0^1([0, 1], \gamma^* TM)$ onto $\ker(P + \mathcal{R}_\gamma)$. Because Π has finite rank, it is bounded with respect to the L^2 norm and therefore extends uniquely to a bounded operator on $L^2([0, 1], \gamma^* TM)$, which is still a projection onto $\ker(P + \mathcal{R}_\gamma)$ (since it is idempotent), but not necessary an orthogonal projection. Set $Q := P + \mathcal{R}_\gamma + \Pi$. Then Q is zeta-admissible by Prop. 5.2 because it can be written in the form $Q = P(\text{id} + W)$ with $W = P^{-1}(\mathcal{R}_\gamma + \Pi)$, which is trace-class by Lemma 2.1. Hence Q is zeta-admissible. With respect to the orthogonal basis f_1, f_2, \dots of the space $H_0^1([0, 1], \gamma^* TM)$ used above, we have

$$(f_i, Q f_j)_{L^2} = \begin{cases} (f_i, f_j)_{L^2} & \text{if } 1 \leq i, j \leq k \\ (f_i, (P + \mathcal{R}_\gamma) f_j)_{L^2} & \text{if } i, j > k \\ 0 & \text{if } 1 \leq i \leq k \text{ and } j > k. \end{cases}$$

To see that third case, if $1 \leq i \leq k$, and $j > k$, calculate

$$(f_i, Q f_j)_{L^2} = (f_i, (P + \mathcal{R}_\gamma) f_j)_{L^2} + (f_i, \Pi f_j)_{L^2} = ((P + \mathcal{R}_\gamma) f_i, f_j)_{L^2} = 0.$$

Hence the infinite matrix with entries $(f_i, Q f_j)_{L^2}$ is block triangular with respect to the orthogonal splitting of $H_0^1([0, 1], \gamma^* TM)$ into $\ker(P + \mathcal{R}_\gamma)$ and its orthogonal complement, and we have

$$\det \left((f_i, Q f_j)_{L^2} \right)_{1 \leq i, j \leq N} = \det \left((f_i, f_j)_{L^2} \right)_{1 \leq i, j \leq k} \det \left((f_i, (P + \mathcal{R}_\gamma) f_j)_{L^2} \right)_{k+1 \leq i, j \leq N}.$$

for all $N > k$. Plugging in (5.4) and (5.5), we then obtain

$$\begin{aligned} \det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2} \det(\text{id}|_\gamma) &= \lim_{N \rightarrow \infty} \det \left((f_i, Q f_j)_{L^2} \right)_{1 \leq i, j \leq N}^{1/2} \\ &= \lim_{N \rightarrow \infty} \det \left((f_i, P^{-1} Q f_j)_{H^1} \right)_{1 \leq i, j \leq N}^{1/2} \\ &= \det^{H^1}(P^{-1} Q)^{1/2}. \end{aligned}$$

Because $P^{-1/2} : L^2([0, 1], \gamma^* TM) \longrightarrow H_0^1([0, 1], \gamma^* TM)$ is an isometry, we obtain

$$\det^{H^1}(P^{-1} Q) = \det^{L^2}(P^{-1/2} Q P^{-1/2}).$$

Again, we have by Prop. 5.2,

$$\det^{L^2}(P^{-1/2} Q P^{-1/2}) \det_\zeta(P^{1/2}) = \det_\zeta(P^{-1/2} Q) = \det_\zeta(P^{1/2}) \det^{L^2}(P^{-1} Q)$$

so that $\det^{L^2}(P^{-1/2} Q P^{-1/2}) = \det^{L^2}(P^{-1} Q)$.

Let now $\tilde{\Pi}$ be the L^2 -orthogonal projection in $L^2([0, t], \gamma^* TM)$ onto $\ker(P + \mathcal{R}_\gamma)$ and set $\tilde{Q} := P + \mathcal{R}_\gamma + \tilde{\Pi}$. We claim that $\det_\zeta(\tilde{Q}) = \det_\zeta(Q)$. To see this, notice first that

$$P + \mathcal{R}_\gamma + \tilde{\Pi} = (P + \mathcal{R}_\gamma + \Pi)(\text{id} + W),$$

where $W = (P + \mathcal{R}_\gamma + \Pi)^{-1}(\tilde{\Pi} - \Pi)$, which is trace-class. Now with respect to the orthogonal splitting of $L^2([0, 1], \gamma^* TM)$ into $\ker(P + \mathcal{R}_\gamma)$ and its orthogonal complement, the operators in question are given by

$$\Pi \cong \begin{pmatrix} \text{id} & * \\ 0 & 0 \end{pmatrix} \quad \tilde{\Pi} \cong \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix} \quad P + \mathcal{R}_\gamma + \Pi \cong \begin{pmatrix} \text{id} & * \\ 0 & P + \mathcal{R}_\gamma \end{pmatrix}.$$

Therefore W is upper triangular with respect to the splitting, hence quasi-nilpotent so that $\det(\text{id} + W) = 1$. Thus by Prop. 5.2, we have

$$\det_\zeta(\tilde{Q}) = \det_\zeta(Q) \det(\text{id} + W) = \det_\zeta(Q).$$

Clearly, the spectrum of \tilde{Q} is the same as the spectrum of $P + \mathcal{R}_\gamma$ except that the k -fold eigenvalue zero is replaced by k times the eigenvalue one. Hence $\zeta_{\tilde{Q}}(z) = \zeta_{P+\mathcal{R}_\gamma}(z) + k$ and $\det_\zeta(\tilde{Q}) = \det'_\zeta(P + \mathcal{R}_\gamma)$. By Prop. 5.2 and Example 5.1, we therefore have

$$\det(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}})^{1/2} \det(d\text{id}) = \det^{L^2}(P^{-1}\tilde{Q})^{1/2} = \frac{\det_\zeta(\tilde{Q})^{1/2}}{\det_\zeta(P)^{1/2}} = \frac{\det'_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)^{1/2}}{\det'_\zeta(-\nabla_s^2)^{1/2}}.$$

Plugging this into (5.3) gives the result. \square

6 Ordinary Differential Equations and the Gel'fand-Yaglom Theorem

It is a well-known fact from Riemannian geometry that for a geodesic $\gamma \in \Gamma_{xy}^{\min}$, we have

$$s \, d \exp_x|_{s\dot{\gamma}_{xy}(0)} = J(s), \tag{6.1}$$

for each $s \in [0, 1]$ where $J(s) \in \text{Hom}(T_{\gamma(0)}M, T_{\gamma(s)}M)$ is the solution to the *Jacobi equation*

$$\nabla_s^2 J(s) = \mathcal{R}_\gamma(s)J(s), \quad J(0) = 0, \quad \nabla_s J(0) = \text{id}. \tag{6.2}$$

see Corollary 1.12.5 in [Kli95] or Thm. II.7.1 in [Cha06]. Hence the Jacobian of the exponential map defined in (1.9) is given by $J(x, y) = \det(J(1))$. Using our results above, we therefore obtain a way to calculate infinite-dimensional determinants by solving an ordinary differential equation.

Theorem 6.1 (Gel'fand-Yaglom). *Let $V_i \in C^\infty([0, t], \mathbb{R}^{n \times n})$, $i = 1, 2$ be functions with values in symmetric matrices and consider the differential operators*

$$P_i := -\frac{d^2}{ds^2} + V_i.$$

Assume that all eigenvalues of P_1 and P_2 are positive. Then we have

$$\frac{\det_{\zeta}(P_2)}{\det_{\zeta}(P_1)} = \frac{\det(J_2(t))}{\det(J_1(t))},$$

where the $J_i(s)$ are the unique matrix-valued solutions of

$$J_i''(s) = V_i(s)J_i(s), \quad J_i(0) = 0, \quad J_i'(0) = \text{id}.$$

It seems that the name of the theorem stems from an older result by Gel'fand and Yaglom [GY60], who express the expectation value of certain Wiener functionals as the solution to an ordinary differential equation, but without mentioning zeta determinants. A proof of Thm. 6.1 can be found in [Kir10] or [KM03] for the scalar case (i.e. $m = 1$), using contour integrals. As we demonstrate below, Thm. 6.1 combined with Prop. 5.5 enables a different proof of the identity

$$\det(\nabla^2 E|_{\gamma_{xy}}) = J(x, y)$$

that gets away without having to calculate the messy term $\Upsilon_{\tau}(\gamma)$. However, this works only in the non-degenerate case. Furthermore, it turns out that the results obtained with our methods (Corollary 4.7 and 5.5) suffice to prove Thm. 6.1.

Proof (of Corollary 4.7, using Thm. 6.1). The vector bundle $\gamma_{xy}^* TM$ over $[0, 1]$ has a canonical trivialization using parallel transport along γ_{xy} , so that Thm. 6.1 is applicable. In this local trivialization, set $V_1(s) \equiv 0$ and $V_2(s) = \mathcal{R}_{\gamma_{xy}}(s)$, the Jacobi endomorphism (2.7) along γ_{xy} . Then use Thm. 6.1 with $P_1 = -\nabla_s^2$ and $P_2 = -\nabla_s^2 + \mathcal{R}_{\gamma_{xy}}$, the Jacobi operator. Clearly, P_1 has only positive eigenvalues, and since $(x, y) \in M \bowtie M$, P_2 has only positive eigenvalues as well (compare Thm. 15.1 in [Mil63]).

Now $J_1(s) = \text{id}$ so that $\det(J_1(1)) = 1$. On the other hand, by (6.1), we have $\det(J_2(1)) = J(x, y)$. Therefore,

$$\det(\nabla^2 E|_{\gamma_{xy}}) = \frac{\det_{\zeta}(-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}})}{\det_{\zeta}(-\nabla_s^2)} = \frac{\det(J_2(1))}{\det(J_1(1))} = \frac{J(x, y)}{1},$$

where we first used Prop. 5.5 and then Thm. 6.1. \square

Proof (of Thm. 6.1, using Corollary 4.7). Since we only calculate the ratio, we may assume $V_1 \equiv 0$. Now given a smooth function $V := V_2$ with values in symmetric $(n \times n)$ -matrices, define on $M = \mathbb{R} \times \mathbb{R}^n$ (equipped with coordinates s, x^1, \dots, x^n) a Riemannian metric as follows. Choose neighborhoods U_1 and U_2 of $[0, t] \times \{0\}$ in M such that $\overline{U_1} \subset U_2$. On U_1 set

$$g_{ss}(s, x) = 1 + V_{ij}(s)x^i x^j, \quad g_{sj}(s, x) = 0, \quad g_{ij}(s, x) = \delta_{ij},$$

where $1 \leq i, j \leq n$ and $V_{ij}(s)$ are the entries of $V(s)$; on the complement on U_2 , set $g_{ss} = 1$, $g_{sj} = 0$, $g_{ij} = \delta_{ij}$; on $U_2 \setminus U_1$, choose a smooth interpolation between the two

metrics. One can choose the open sets and the interpolation in such a way that the resulting metric is non-degenerate; then M becomes a complete Riemannian manifold. The curve $\gamma(s) := (s, 0, \dots, 0)$ is a geodesic from $x := (0, \dots, 0)$ to $y := (t, 0, \dots, 0)$, because all Christoffel symbols vanish at points in $[0, t] \times \{0\}$, as is easy to calculate. It is the unique shortest geodesic between x and y if and only if the Jacobi operator $-\nabla_s^2 + \mathcal{R}_\gamma$ on $[0, t]$ has only positive eigenvalues (see [Mil63, Thm 15.1]), which we assume from now on. On the other hand, one can easily compute that the Jacobi endomorphism (2.7) is explicitly given by

$$\mathcal{R}_\gamma(s) = \begin{pmatrix} 1 & 0 \\ 0 & V(s) \end{pmatrix}, \quad (6.3)$$

so that by (6.1), the differential of the exponential map is given by

$$d\exp_x|_{s\dot{\gamma}(0)} = \frac{1}{s} \begin{pmatrix} 1 & 0 \\ 0 & J_2(s) \end{pmatrix},$$

where $J_2(s)$ is the unique matrix solution of

$$J_2''(s) = V(s)J_2(s), \quad J_2(0) = 0, \quad J_2'(0) = \text{id}.$$

The shortest geodesic travelling from x to y in time one, on the other hand, is given by $\gamma_{xy}(s) = \gamma(st)$. Hence

$$J(x, y) = \det(d\exp_x|_{\dot{\gamma}_{xy}(0)}) = \det(d\exp_x|_{t\dot{\gamma}(0)}) = \frac{\det(J_2(t))}{t^{n+1}} = \frac{\det(J_2(t))}{\det(J_1(t))},$$

where $J_1 = t \text{id}$ is the matrix solution of the equation $J_1''(t) = 0$ with initial conditions $J_1(0) = 0$, $J_1'(0) = \text{id}$. By Prop. 5.5 and Corollary 4.7, we therefore have

$$\frac{\det_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)}{\det_\zeta(-\nabla_s^2)} = \frac{\det_\zeta(-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}})}{\det_\zeta(-\nabla_s^2)} = J(x, y) = \frac{\det(J_2(t))}{\det(J_1(t))},$$

where we also used that the quotient on the left hand side does not depend on s as is easy to verify by considering the eigenvalues. Finally, because of (6.3), the bundle separates into the direction tangent to $\dot{\gamma}$ and the orthogonal directions, so we obtain

$$\frac{\det(J_2(t))}{\det(J_1(t))} = \frac{\det_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)}{\det_\zeta(-\nabla_s^2)} = \frac{\det_\zeta(P_2) \det_\zeta(-\partial_s^2)}{\det_\zeta(P_1) \det_\zeta(-\partial_s^2)} = \frac{\det_\zeta(P_2)}{\det_\zeta(P_1)}.$$

This finishes the proof of Thm. 6.1. \square

Remark 6.2. Of course, in the formulation of Thm. 6.1, one could use the Fredholm determinant of $P_1^{-1}P_2$ instead of the quotient of the zeta determinants. This way, one would get away without having to use Prop. 5.5. That is, Thm. 6.1 can also be written as a theorem about usual Fredholm determinants.

There is a Gel'fand-Yaglom-type theorem for the degenerate case too. As in Section 3 of [KM03], one proves the following result.

Theorem 6.3 (Degenerate Gel'fand-Yaglom). *With notations as in Thm. 6.1, assume that P_2 is a positive operator, but that P_1 has the eigenvalue zero. Then we have*

$$\frac{\det'_\zeta(P_2)}{\det_\zeta(P_1)} = \frac{\det\left(\int_0^1 J_2(s)^* J_2(s) ds\right)}{\det(J_1(1)) \det(J'_2(1))}.$$

This can be used to prove the following formula for the lowest term in the heat expansion, which only depends on the Riemannian exponential map.

Theorem 6.4. *For $x, y \in M$, set $S_{xy} := \{\dot{\gamma}(0) \mid \gamma \in \Gamma_{xy}^{\min}\} \subset T_x M$. Under the assumptions of Thm. 1.2, S_{xy} is a k -dimensional submanifold of $T_x M$ and we have*

$$\lim_{t \rightarrow 0} (4\pi t)^{k/2} \frac{p_t^L(x, y)}{e_t(x, y)} = \int_{S_{xy}} [\gamma_v \|_0^1]^{-1} \det(J'(1))^{1/2} dv,$$

where for $v \in S_{xy}$, γ_v is defined by $\gamma_v(s) = \exp_x(sv)$, $J(s)$ is given by (6.1) and we integrate with respect to the submanifold measure induced on S_{xy} by the metric on $T_x M$.

Notice that $J(s)$ depends on the underlying geodesic γ , even though this is not reflected in the notation.

Remark 6.5. The formula of Thm. 6.4 should be compared with the formula

$$\lim_{t \rightarrow 0} \frac{p_t^L(x, y)}{e_t(x, y)} = [\gamma \|_0^1]^{-1} \det(J(1))^{-1/2}, \quad (6.4)$$

which holds in the case that $(x, y) \in M \bowtie M$, by (3.4) and (6.1).

Proof. Using Thm. 6.3 on the formula from Thm. 1.2 with $P_2 = -\nabla_s^2 + \mathcal{R}_\gamma$ and $P_1 = -\nabla_s^2$, we obtain

$$\int_{\Gamma_{xy}^{\min}} [\gamma \|_0^1]^{-1} \frac{\det_\zeta(-\nabla_s^2)^{1/2}}{\det'_\zeta(-\nabla_s^2 + \mathcal{R}_\gamma)^{1/2}} d^{L^2} \gamma = \int_{\Gamma_{xy}^{\min}} [\gamma \|_0^1]^{-1} \frac{\det(J'(1))^{1/2}}{\det\left(\int_0^1 J(s)^* J(s) ds\right)^{1/2}} d^{L^2} \gamma,$$

since we have $J_1(s) = \text{id}$, hence $\det(J_1(1)) = 1$, and $J_2(s) = J(s)$, given by (6.1). Define the map

$$\phi : S_{xy} \longrightarrow \Gamma_{xy}^{\min}, \quad v \longmapsto \gamma_v,$$

where $\gamma_v(s) = \exp_x(sv)$. Then by the transformation formula, the integral above is given by

$$\int_{S_{xy}} [\gamma_v \|_0^1]^{-1} \frac{\det(J'(1))^{1/2}}{\det\left(\int_0^1 J(s)^* J(s) ds\right)^{1/2}} \det(d\phi|_v) dv \quad (6.5)$$

Fix $v \in S_{xy}$. For an orthonormal basis e_1, \dots, e_n of $T_x M$, let X_1, \dots, X_n be the Jacobi fields along γ_v with $\nabla_s X_j(0) = e_j$. Then $J_1(s) = (X_1(s), \dots, X_n(s))$ and

$$\det \left(\int_0^1 J_1(s)^* J_1(s) ds \right) = \det \left(\int_0^1 \left(\langle X_i(s), X_j(s) \rangle \right)_{1 \leq i, j \leq n} ds \right) = \det \left((X_i, X_j)_{L^2} \right)_{1 \leq i, j \leq n}.$$

Similarly,

$$\det(d\phi|_v) = \det \left((d\phi|_v e_i, d\phi|_v e_j)_{L^2} \right)_{1 \leq i, j \leq n} = \det \left((X_i, X_j)_{L^2} \right)_{1 \leq i, j \leq n}. \quad \square$$

Therefore, two of the determinants in (6.5) cancel and we are left with the integrand from the theorem.

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